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Higher order stabilized time stepping in unfitted finite element method on moving domains

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I hereby declare that I have written this thesis independently and without the use of documents or aids other than those stated. I have mentioned all used sources and cited them correctly according to established academic citation rules.

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Abstract

Based on a new finite element method, which is introduced in [1], for the numerical solutions of partial differential equations posed on moving domains, this thesis extends the stabilized implicit Euler method in time to some higher order stabilized mehtods in time. A standard geometrically unfitted finite element method with a stabilization term is still using for space discretization. This thesis includes a stability analysis on the full discretization mehtod. The numerical example demonstrate the practical efficiency of the higher order methods.

Keywords : moving domains, higher order, unfitted finite element method.

Contents

Chapter 1

Introduction

Partial differential equations(PDEs) arise in a wide variety of sciences and engineering applications. Many cases are PDEs posed on moving domains, we may call theses cases as moving domain problems. This remains a challenge within the general subject of numerical simulation of PDEs. One example is the simulation of two cells mergeing into one big cell. The interfaces that separate these two cells may have large deformations and even topology changes when they are merging. Hence, the essential framework of moving domain problems is geometric since the domains on which PDEs posed change all the time. Hence, relative to static domain problems, moving domain problems are more complicated. The efficient and accurate simulations of moving domain problems require the development of advanced numerical techniques.

The finite element method(FEM), cf. Figure (1.1(a)), is a standard numerical method for solving PDEs by approximating continuous quantities as a linear combination of discrete basis functions, regularly spaces the domain into so-called elements and employs a mesh fitted to the geometry i.e. the boundary of the physical domain coincides with the boundaries of the elements. Although FEM can be applied to the problems of complex geometry, when the geometry is evolving with strong deformation, keeping the mesh conforming to the geometry may lead to significant complications. Thus, unfitted finite element method (unfitted FEM), cf. Figure (1.1(b)) is receiving rapidly increasing interest in recent years. Relative to FEM uses a fitted mesh, unfitted FEM uses an unfitted mesh which means the mesh is independent of the geometry i.e. the boundary of the physical domain may not coincide with the boundaries of the elements. Therefore, it simplifies the construction of numerical methods for the moving domain problems that exhibit strong deformation or even topology changes.

In unfitted FEM method, level set methods are used to describe the geometry. The level set function make it easy to represent the shapes that are changing and provides numerical computations without having to parameterize the objects.

Since in the unfitted FEM method the mesh is not fitted to the geometry, an implementation of such method thus requires an evaluation of integrals on the so-called cut elements [2]. Cutting the elements can lead to elements with very small intersections, or we say small cut elements, with the boundary of domain. Such small cut elements may cause ill-conditioning when we compute numerical solution of PDEs and prohibit the application of a whole set of inverse inequalities. To this end, a stabilization term is often added to control the discrete functions on small cut elements by close-by neighbors with large intersection to overcome the ill-conditioning and stability problems.

In [1] and [3], a new FEM to solve numerical solution of PDEs posed on moving domains has been proposed and analyzed. Instead of using a space-time variational framework, the time derivatives are discretized by finite difference approximation and a standard geometrically unfitted finte element with a stabilization term is used to accommodate spacial variations. This method, unlike space-time Galerkin methods, does not require a reconstruction of physical domain on each time slab.

Based on the results from [1] and [3], in this thesis we introduce some higher order mehtods for time discretization. We also give three characterizations of tripazoidal rules for moving domains and a midpoint rule for moving domains. A geometrically unfitted FEM with a stabilization term which is introduced in [1] is used for the spacial discretization. The goal of this thesis is to construct a fully discrete method with higher order stabilized time stepping for PDEs posed on moving domains.

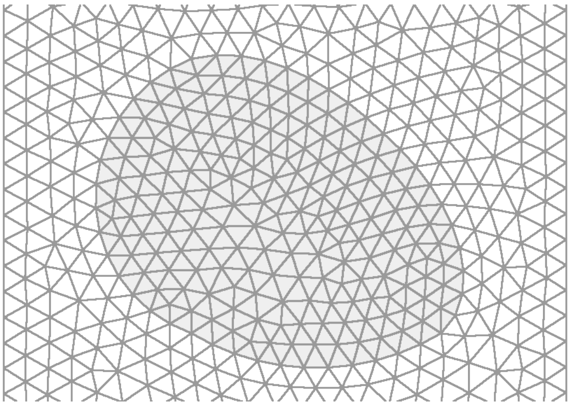
1.1 Outline of the thesis

The outline of the thesis is given as follows:

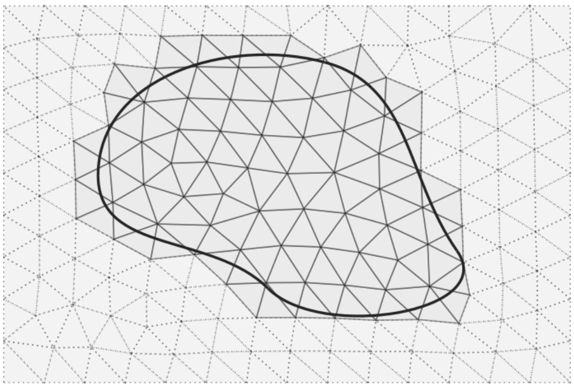
- In Chapter 2 we describe a mathematical model of PDEs posed on time-dependent domains and the fundamental numerical techniques for solving this type of problem.
- In Chapter 3, with an extension operator we discuss some higher order semi-disretizations for moving domains. We extend the implicit Euler time stepping method to BDF2 and BDF3 time stepping methods. We introduce an explicit Euler semi-discretization, and combine with the implicit Euler semi-discretization we introduce three characterizations of trapezoidal rules for moving domains and a midpoint rule for moving domains.
- In Chapter 4, in the full discretization of the methods introduced in chapter 3, we combine the solution with a stabilization term.
- In Chapter 5, we present the stability analysis of a full discretization based on stabilized BDF2 time stepping mehtod, a full discretization based on stabilized explicit time stepping method and first and second characterizations of tripazoidal rules.
- Chapter 6, we presents the numerical experiments of the implicit Euler stabilized time stepping method, BDF stabilized time stepping methods and the explicit Euler time stepping

method.

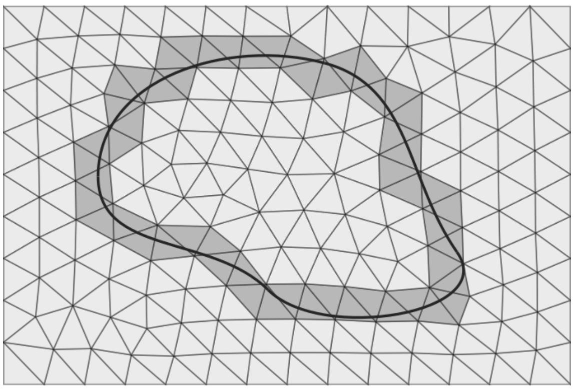
- Conclusion in Chapter 7 with a summary of the thesis. Open problems will be discussed and we propose some directions for further research.



(a) Mesh fits PDEs domain. The boundary of physical domain coincides with the boundaries of the elements. The mesh is called fitted mesh.



(b) Mesh contains PDEs domain. The boundary of physical domain does not coincide with the boundaries of the elements. The mesh is called unfitted mesh.



(c) Elements cutted by the boundary of domain. The thick color elements are elements cutted by the boundary of domain.

Figure 1.1: Finite elements. Sketch(a) illustrates fitted finite elements. Sketch(b) illustrates unfitted finite elements. Sketch(c) illustrates the elements which are cutted by the boundary.

Chapter 2

Mathematical model and fundamental numerical techniques

In this chapter, we describe a mathematical model of PDEs posed on moving domains. Further, we briefly introduce the numerical techniques which are used to solve this type of problem.

2.1 Mathematical model

2.1.1 Notation

We define $\Omega_t := \Omega(t)$, $\Gamma_t := \Gamma(t) = \partial\Omega_t$. We are given a sufficently regular time-dependent domain

$$\Omega_t \subset \mathbb{R}^d, \quad d = 2, 3, \quad t \in [0, T],\ T > 0. \tag{2.1}$$

We assume that Ω_t moves smoothly for all $t \in [0, T]$. Suppse there is a one-to-one continuous mapping which is introduced in [1]

$$\Psi(t) : \Omega_0 \to \Omega_t, \quad \forall t \in [0, T] \tag{2.2}$$

from the reference domain $\Omega_0 \subset \mathbb{R}^d$. For all $t \in [0, T]$, the time-dependent domain Ω_t together with its neighborhood

$$\mathcal{O}_\delta(\Omega_t) := \{\mathbf{x} \in \mathbb{R}^d : \text{dist}(\mathbf{x}, \Omega_t) \leq \delta\}. \tag{2.3}$$

are contained in a polygnoal background domain $\tilde{\Omega}$, cf. Figure 2.1,where

$$\delta = c_\delta \mathbf{w}^{\mathbf{n}}_\infty \Delta t,\ c_\delta > 0. \tag{2.4}$$

with $\mathbf{w}^{\mathbf{n}}_\infty := \max_{t\in[0,T)} \|\mathbf{w} \cdot \mathbf{n}\|_{L^\infty(\Gamma_t)}$.

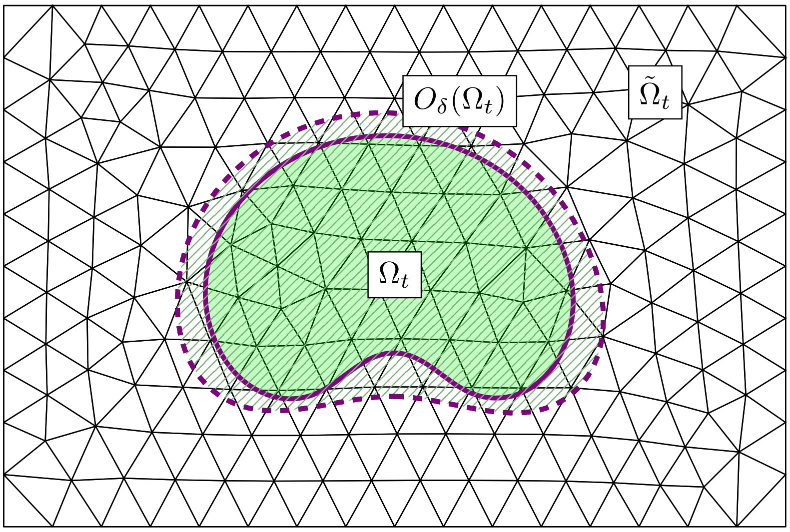


Figure 2.1: δ-neighborhood $\mathcal{O}_\delta(\Omega_t)$ of Ω_t

2.1.2 Mathmatical model

The mathematical model is taken from [1]. Let $\mathbf{w} : \Omega_t \to \mathbb{R}^d$ be the material velocity of the particles from Ω_t, then the continuous mapping Ψ_t can be defined as a *Lagrangian* mapping from Ω_0 to Ω_t, i.e. $\forall y \in \Omega_0$, $\Psi(t, y)$ solves the ODE system

$$\begin{aligned} \Psi_0(y) &= y, \\ \frac{\partial \Psi_t(y)}{\partial t} &= \mathbf{w}(t, \Psi_t(y)),\ t \in [0, T] \end{aligned} \tag{2.5}$$

Then the transient convection-diffusion equation in conservative form in Ω_t is given by

$$\frac{\partial u}{\partial t} + \mathrm{div}(u\mathbf{w}) - \nu \Delta u = f, \quad t \in [0, T] \tag{2.6}$$

where $\nu > 0$ is the constant diffusion coefficient and $\mathbf{w}$ is the velocity field. Let $\mathbf{n}$ be the unit normal on the boundary $\Gamma_t := \partial \Omega_t$. In this thesis we assume that the flux is zero on Γ_t i.e.

$$\nabla u \cdot \mathbf{n} = 0 \quad on\, \Gamma_t,\ t \in (0, T]. \tag{2.7}$$

Then the time-dependent convection-diffusion problem is complemented by the governing equation (2.6), the initial condition $u(\mathbf{x}, 0) = u_0(\mathbf{x})$ at time t=0 and flux boundary condition (2.7), cf. Figure 2.2 for an example.

Remark 1. *(2.7) is a suitable boundary condition for the conservation of u which can be obtained by applying Reynold's transport theorem for moving domains.*

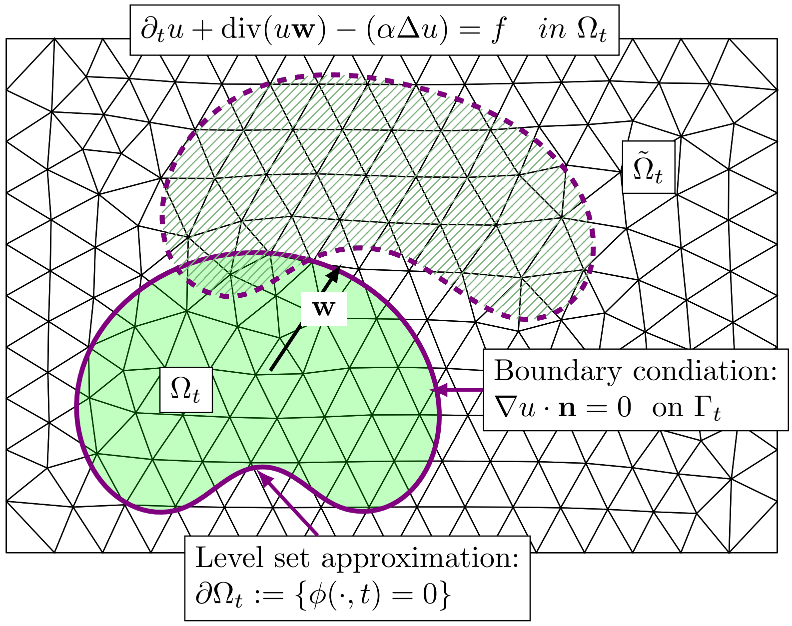


Figure 2.2: Mathematical model of PDEs posed on moving domain. $\tilde{\Omega}_t$ is a polynomial background domain. Ω_t is a time-dependent domain which is approximated by a level set $\{\phi(\cdot,t)\}$ and moves smoothly in the $\mathbf{w}$ velocity field. Γ_t can be described by $\{\phi(\cdot,t) = 0\}$. The flux is zero on Γ_t

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2.2 Numerical setting for the space discretization

The numerical setting for the moving domains in this thesis is based on an unfitted FEM for the space discretization, a representation of the geometry by a level set and time discretization by methods based on finite differences.

2.2.1 Unfitted discretization in space

Motivation

Standard FEM is usually a fitted discretization method which means that the mesh fits the PDEs domain. Relative to standard FEM, unfitted FEM means that mesh contains the PDEs domain, cf. Figure 1.1. The advantages of unfitted FEM is that it avoids remeshing when strong deformations or topology changes occur and mesh generation for complex geometries can be much cheaper.

Construction of unfitted finite element spaces

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape regular triangulation of $\tilde{\Omega}$ consisting of simplexes T_s with a characteristic mesh size $h := \max\{h_{T_s} | T_s \in \mathcal{T}_h\}$, cf Figure 2.3. In Figure 2.3 we observe that the boundary Γ_t does not coincide with elements boundaries, so the triangulation is *unfitted*.

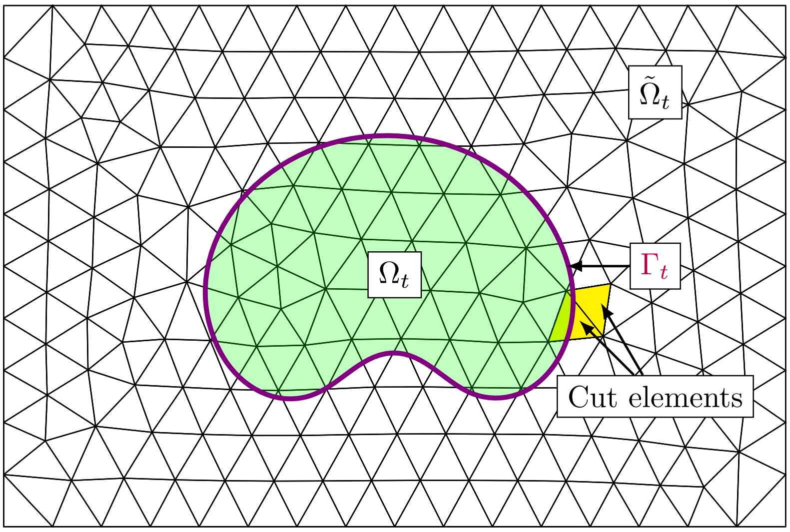

Figure 2.3: Unfitted triangulation. The background domain $\tilde{\Omega}$ is covered by a triangulation $\{\mathcal{T}_h\}_{h>0}$. The triangulation $\{\mathcal{T}_h\}_{h>0}$ contains the physcial domain Ω_t. The boundary Γ_t does not coincide with the elements boundaries, so the triangulation is unfitted. The yellow elements are example elements which are cutted by the boundary Γ_t

The elements $T_{\Gamma_t,s}$ with $\Gamma_t \bigcap T_s \neq \emptyset$ are called cut elements. We introduce some notation for *unfitted* triangulation. We define $\Omega^n := \Omega_{t_n}$, $\Gamma^n := \Gamma_{t_n}$, $n = 0, 1, ..., N$. $\forall T_s \in \mathcal{T}_h$, we denote $T_s^n := T_s \bigcap \Omega^n$ the part of T_s in Ω^n and $\Gamma_{T_s}^n := T_s \bigcap \Gamma^n$ the part of the boundary lies in T_s. We denote $\Gamma_{\Gamma,s}^n := \{T_s : T_s \bigcap \Gamma^n \neq \emptyset\}$ the set of elements that are cutted by the boundary.

In semi-discretization method, we combine the solution with its extension in each time step. In full discretization method, the extension can be realized by a stabilization term which is used to combine the solution and the extension on a discretely extended domain. We can choose a suitable δ_h so that Ω_h^{n+1} is contained in the extended domain $\mathcal{O}_{\delta_h}(\Omega_h^n)$. To this end, we consider a *active mesh*, cf. Figure 2.4 which is defined in a slightly simpler version in [1]. It is the set of all elements that have some parts in the extended domain,

$$\begin{aligned} \mathcal{T}_\delta^{n,l} &:= \{\mathcal{S} \in \mathcal{T}_h : \mathrm{dist}(\mathbf{x}, \Omega_h^n) \leq l \cdot \delta_h \text{for some } \mathbf{x} \in \mathcal{S}\}, \\ \mathcal{O}_{\delta_h,\mathcal{T}}^{n,l} &:= \{\mathbf{x} \in \mathcal{S} : \mathcal{S} \in \mathcal{T}_\delta^{n,l}\}. \end{aligned} \tag{2.8}$$

Here $l \in \mathbb{N}$ indicates the number of time steps in which the domains $\Omega_h^n, \Omega_h^{n+1}, \ldots, \Omega_h^{n+l}$ are still contained in the domain corresponding to $\mathcal{T}_\delta^{n,l}$. On these (extended) active meshes, we construct the unfitted finite element spaces

$$V_h^{n,l} := \{v \in C(\mathcal{O}_{\delta_h,\mathcal{T}}^{n,l}) : v \in \mathbb{P}_k(\mathcal{S}), \forall \mathcal{S} \in \mathcal{T}_h^{n,l}\}, \quad k \geq 1. \tag{2.9}$$

These spaces are the restricions of the time-independent bulk space V_h on all simplices from $T_\delta^{n,l}$.

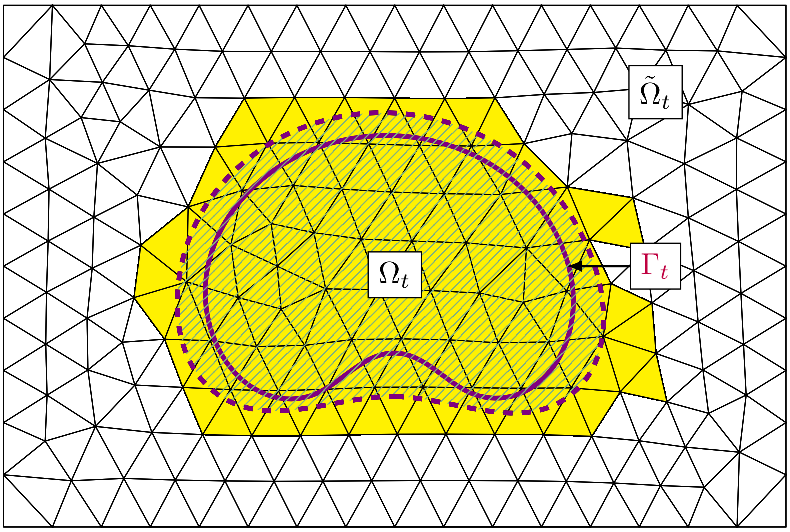

Figure 2.4: Active mesh. The yellow mesh is the active mesh at time t on which we define the unfitted finite element space $V_h^n = V_h^{n,1}$.

If no index l is used, we assume $l = 1$, i.e. we extend the domain only for the next time step and define $\mathcal{T}_\delta^n := \mathcal{T}_\delta^{n,1}$, $\mathcal{O}_{\delta_h,\mathcal{T}}^n := \mathcal{O}_{\delta_h,\mathcal{T}}^{n,1}$ and $V_h^n := V_h^{n,1}$.

2.2.2 Representation of geometry

Level set methods are a conceptual framework for using level sets as a tool for describing surfaces and shapes. The advantage of the level set model is that one can perform numerical computations by the Euclidian approach [4]. Also, the level set method makes it convenient to follow shapes that change topology, for example a cell spliting in two cells or two cells merging into a cell. All these make level set method a suitable tool for modeling time-varying objects of moving domain problems.

Level set methods amount to representing a closed curve using a function ϕ which is called level set function. In moving domain problems, this closed curve is the boundary of the moving domain Ω_t defined as $\Gamma_t := \partial\Omega_t$. It can be represented as the zero level set of ϕ by

$$\Gamma_t = \{(\phi(\cdot, t) = 0\}, \quad t \in [0, T]. \tag{2.10}$$

The level set method describes the boundary Γ_t implicitly through the level set funtion ϕ. The function ϕ takes positive value outside the region delimited by Γ_t and negative values inside. See Figure 2.5 for an example.

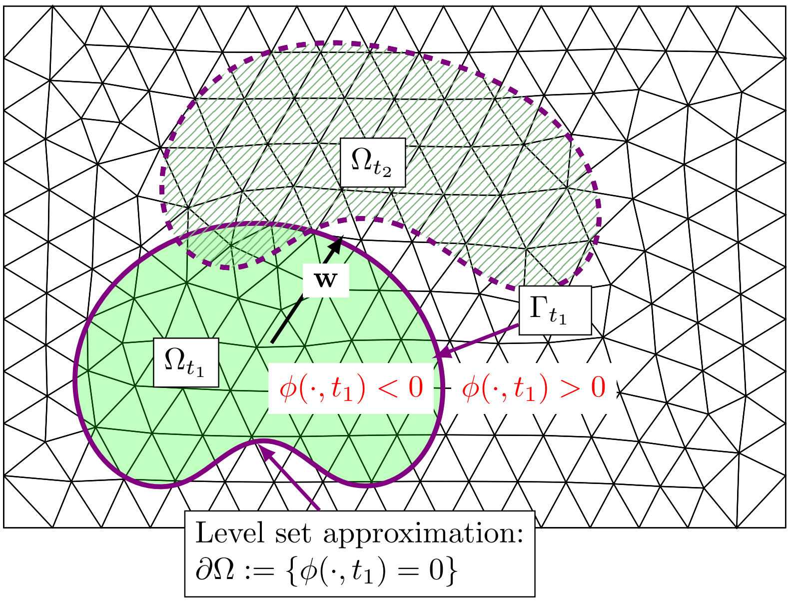

Figure 2.5: An example of geometry description by level set method.

2.3 Basic finite difference method for time integration of ODEs

2.3.1 Motivation

For a moving domain problem, the system of equations generated by spacial discretization is not a system of ODEs. So we can not directly apply standard numerical methods to time discretization. The modified numerical methods for time discretization will be introduced in next chapter. Before that, we recall some basic finite difference methods for time derivatives.

2.3.2 Ordinary Differential Equations

First-order ODE system can typically be written in explicit form

$$u'_t = f(t, u_t) \tag{2.11}$$

where $f : [t_0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$, $u : [t_0, \infty) \to \mathbb{R}^d$. If the value of $u(t_0) = u_0 \in \Omega$ is given, then the pair of equations

$$u'_t = f(t, u_t), \quad u_{t_0} = u_0 \tag{2.12}$$

is known as an *initial value problem*.

Remark 2. *Numerical methods for solving first-order initial value problems often fall into one of two large categories: linear multistep methods or Runge-Kutta methods. A further division can be realized by dividing methods into those that are explicit and those that are implicit.*

2.3.3 Finite difference based methods

The Euler methods

To discretize the ODE in time with $t \in [0, T]$, the time interval is discretized by a temporal grid

$$0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = T. \tag{2.13}$$

The standard finite difference discretization of the time derivative is given by

$$\frac{u^{n+1} - u^n}{\Delta t} = \lambda f(u^{n+1}, t_{n+1}) + (1 - \lambda) f(u^n, t_n),\ 0 \leq \lambda \leq 1 \tag{2.14}$$

where $\Delta t = t_{n+1} - t_n$ be the time step size and λ is the implicitness parameter with

$$\begin{aligned} &\lambda = 0,\ \text{the forward Euler method with local truncation error } \mathcal{O}(\Delta t^2) \\ &\lambda = \frac{1}{2},\ \text{Crank-Nicolson method with local truncation error } \mathcal{O}(\Delta t^3) \\ &\lambda = 1,\ \text{the backward Euler method with local truncation error } \mathcal{O}(\Delta t^2) \end{aligned} \tag{2.15}$$

Remark 3. *The forward Euler method is an example of an explicit method with first order of accuracy. This means that the new value u^{n+1} is defined in terms of things that are already known, like u^n. The backward Euler method is an implicit method with first order of accuracy, meaning that we have to solve an equation to find u^{n+1}. Crank-Nicolson method is an average step of the forward and the backward Euler method with second order of accuracy.*

Backward differentation formulas (BDF methods)

The backward difference is a finite difference defined as

$$\delta_n u \equiv \delta u^n \equiv u^n - u^{n-1}. \tag{2.16}$$

where δ denotes the backward difference operator. Repeating the operation of the backward difference operator, we obtain the higher order differences

$$\begin{aligned} \delta_n^2 u &= \delta(\delta u^n) = \delta(u^n - u^{n-1}) \\ &= \delta u^n - \delta u^(n-1) \\ &= (u^n - u^{n-1}) - (u^{n-1} - u^{n-2}) \\ &= u^n - 2u^{n-1} + u^{n-2} \end{aligned} \tag{2.17}$$

The differentiation of Newton's interpolation fomular which is introduced in [5] leads to the equation

$$\sum_{j=1}^{k} \sigma_j \delta^j u^{n+1} = \Delta t f^{n+1} \tag{2.18}$$

where Δt denotes the time step and σ_j are coefficients generated by Taylor extension. Here, we expand (2.18) by only choosing $k = 1, 2, 3$ and obtain the BDF schemes:

$$\begin{aligned} u^{n+1} - u^n &= \Delta t f^{n+1}, \\ \frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1} &= \Delta t f^{n+1}, \\ \frac{11}{6}u^{n+1} - 3u^n + \frac{3}{2}u^{n-1} - \frac{1}{3}u^{n-2} &= \Delta t f^{n+1}. \end{aligned} \tag{2.19}$$

Remark 4. *It is well known that only first- and second-order BDF schemes are A-stable. The definition of A-stability is given in [6].*

Definition 2.3.1 (A-stability)**.** *A k-step method is called A-stable, if all solutions of of (2.20) tend to zero, as $n \to \infty$, when the method is applied with fixed positive* Δt *to any differential equation of the form*

$$u' = cu, \; u(0) = u_0 \in \mathbb{C} \tag{2.20}$$

where c is a complex constant with negative real part.

Remark 5. *Let p be polynomial order of temporal discretization. An A-stable multistep method must be of order $p \leq 2$. More details on A-stability are available in [6] and [7].*

Stability of BDF methods

The stability regions for the BDF methods of order 1 to 6 are shown in Figure 2.6. The stable regions are the exterior of the contours indicated. Note that BDF methods of order 1 to 6 are suitable for stiff equations, since they are stable along the whole of the negative real axis. The contour of the BDF method of order 7 crosses the negative real axis, making it and any higher order BDF method of no value, cf Figure 2.6.

Remark 6. *The second-barrier of Dahlquist limits the utility of the BDF schemes for order great than 3 as a general purpose scheme. References [8] and [9] show that we can solve many computational fluid dynamics problem using BDF3 scheme but still face numerical instabilities when simulating unsteady problems. The reference [10] gives an idea of constructing an optimized, 2nd order, backward differencing formulation BDF2OPT by a linear combination of BDF2 and BDF3 four time levels schemes or BDF2, BDF3 and BDF4 five time level schemes, with an error constant half as large as the conventional BDF2 scheme.*

Summary

So far we described a mathematical model of PDEs posed on moving domains and introduce the numerical settings for this type of problem. Further, we recall some classical finite difference-

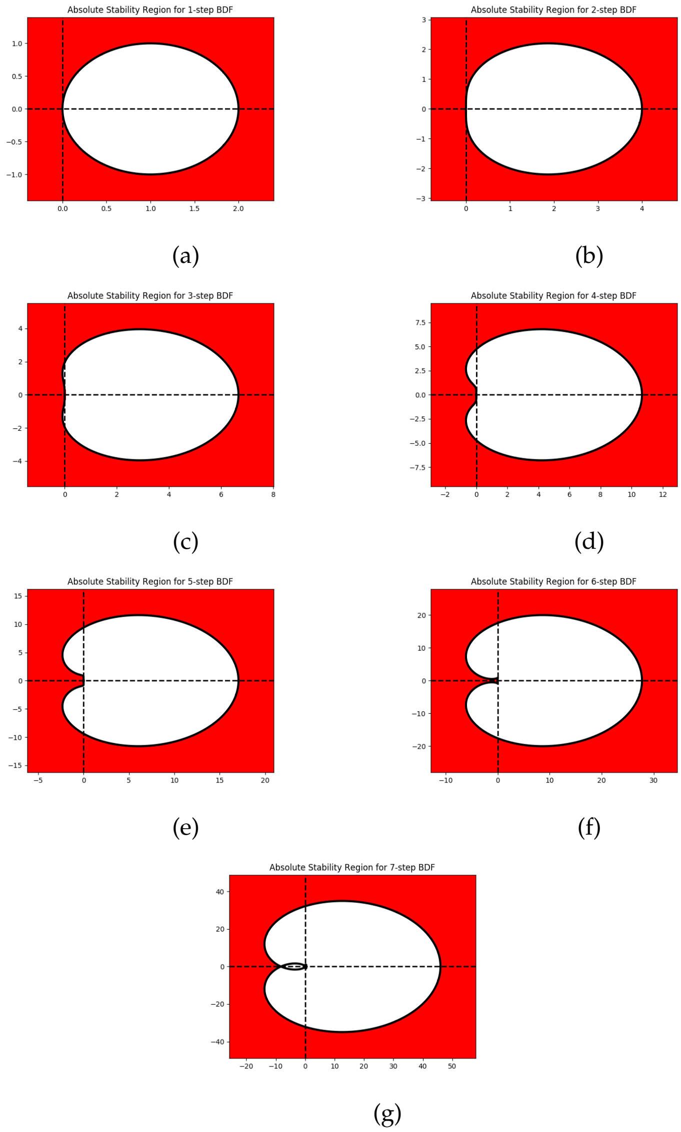

Figure 2.6: Stability region of BDF methods. The plot are generated using the Nodepy python package [11]

based methods for ODE systems such as the Euler methods and BDF methods. Since the system generated by spacial discretization is not a system of ODEs for moving domain problems in an unfitted setting, in order to compute numerical solution we introduce a spacial extension operator and some modified finite difference methods in next chapter.

Chapter 3

Semi-discretization method

In this chapter we apply some of the basic ODE integrators to time integration of moving domain problems. The remain challenge is that the system of equations which are generated by spacial discretization are not systems of ODEs. To this end, we introduce a modified numerical mehtod for time discretization. The method is based on the fundamental result of the existence of continuous extension operators in Sobolov spaces. The result allows to identify the solution to the PDE with its smooth extension and further to design a finite element method, which solves at each discrete time instance for the extended solution in the computational domain.

Using the continuous spacial extension operator $\mathcal{E}$ we can deal with time derivatives by finite differences in the physical domain at least within one time step. We add a stabilization term to the unfitted finite element formulation which yields a numerical approximation to the extension operator without any explicit extension step. This stabilization term acts in a narrow band containing the physical domain boundary.

3.1 A continuous spacial extension operator $\mathcal{E}$

In chapter 2 we define a δ-neighborhood $\mathcal{O}_\delta(\Omega^n)$ of Ω^n, cf. Figure 2.1. $\mathcal{O}_\delta(\Omega^n)$ is set to be large enough so that

$$\Omega^{n+1} \subset \mathcal{O}_\delta(\Omega^n), \quad n = 0, 1, ..., N. \tag{3.1}$$

The continuous spacial extension operator $\mathcal{E}$

Suppose $\Omega \in \mathbb{R}^d$ with *Lipschitz* boundary, the Sobolev space $\mathbf{W}^{1,p}(\Omega),\ 1 \leq p \leq \infty$, consists of all $u \in L^p(\Omega)$ with $\nabla u \in L^p(\Omega)$. It is a *Banach* space equipped the norm

$$\|u\|_{\mathbf{W}^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \tag{3.2}$$

When p=2, it is a *Hilbert* space equipped the norm

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \tag{3.3}$$

We call Ω is a $\mathbf{W}^{1,p}$*-extension domain* with a bounded linear operator

$$\mathcal{E} : \mathbf{W}^{1,p}(\Omega) \to \mathbf{W}^{1,p}(\mathbb{R}^d) \text{ such that } \mathcal{E}u|_\Omega = u|_\Omega \tag{3.4}$$

Remark 7. *Domains with Lipschitz boundary are uniform domains, in [12] Jone Peter.W shows that uniform domains are* $\mathbf{W}^{1,p}$*-extension domains.*

In order to define the extension operator for the time dependent domain Ω^n with *Lipschitz* boundary to its neighborhood $\mathcal{O}(\Omega^n)$, we can assume that the initial domain Ω^0 is a $\mathbf{W}^{1,p}$*-extension domain* with the bounded linear operator

$$\mathcal{E}_0 : \mathbf{W}^{1,p}(\Omega^0) \to \mathbf{W}^{1,p}(\mathbb{R}^d) \quad \text{s.t.} \quad \mathcal{E}_0 u|_{\Omega^0} = u|_{\Omega^0} \tag{3.5}$$

and define a corresponding extension by transformation

$$\mathcal{E}u_t := (\mathcal{E}_0(u_t \circ \Psi_t)) \circ \Psi_t^{-1}, \quad \forall t \in (0, T]. \tag{3.6}$$

In [1], C. Lehrenfeld and M.A. Olshanskii give a detailed description to show the existence of the extension operator $\mathcal{E}$ and complete proofs [[1], Section 3.22] of numerical stability of $\mathcal{E}$. Using this extension operator, we can obtain the semi-discretizations for PDEs posed on moving domain.

Remark 8. *The extension operator* $\mathcal{E}_{n.k}$ *means the extension layer has size* $k \cdot \delta$ *and starts from* Ω^n*. If* $k = 1$ *we skip the index, i.e.* $\mathcal{E}_n = \mathcal{E}_{n,1}$*.*

Lemma 1. *Assume the extension operator* $\mathcal{E} : H^1(\Omega) \to H^1(\mathcal{O}_\delta(\Omega))$ *is continuous. Then it can be identified with a continuous extension operator* $\mathcal{E}^* : H^{-1}(\Omega) \to H^{-1}(\mathcal{O}_\delta(\Omega))$*.*

Proof. Let $g \in H^{-1}(\Omega)$, then there is $u_g \in H^1(\Omega)$ with $g(v) = (u_g, v)_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$ (Riesz). We define $u_g^* := \mathcal{E}u_g$ and define $(\mathcal{E}^* g)(v) := (\mathcal{E}u_g, v)_{H^1(\mathcal{O}_\delta(\Omega))}$, $v \in H^1(\mathcal{O}_\delta(\Omega))$. As the Riesz representation is an isomorphism continuity follows directly. □

3.2 Stabilized time stepping based on implicit methods

For simplicity, we use uniform time steps $\Delta t = \frac{T}{N}$, $t_n = n\Delta t$, n=$0, 1, ..., N$. We define the time inteval as $I_n := [t_n, t_{n+1})$ and the boundary at time step t_n as $\Gamma^n := \Gamma_{t_n}$.

3.2.1 Stabilized time stepping based on the implicit Euler method

In this method we combine the numerical solution for u^n with its extension on $\mathcal{O}_\delta(\Omega^n)$ in each time step by a continuous spacial extension operator $\mathcal{E}_n : H^1(\Omega^n) \to H^1(\mathcal{O}_\delta(\Omega^n))$ which extends funcions on Ω^n to functions on $\mathcal{O}_\delta(\Omega^n)$. This setup guanrantees that u^{n-1} is well-defined on Ω^n. To be more precise, an implicit Euler semi-discrete scheme consists of sampling (2.6) at t_n and approximating the time derivatives by a backward difference

$$D^-_{n,t,\mathcal{E}_{n-1}} u := \frac{u^n - \mathcal{E}_{n-1} u^{n-1}}{\Delta t},\ n{=}1, 2, ..., N. \tag{3.7}$$

The approximation (3.7) turns (2.6) into a differential equation which is discrete in time.

With (3.7) we can obtain the semi-discretization as

$$D^-_{n,t,\mathcal{E}_{n-1}} u := \frac{u^n - \mathcal{E}_{n-1} u^{n-1}}{\Delta t} = Au^n + f^n,\ n{=}1, 2, ..., N \tag{3.8}$$

holds in $H^{-1}(\Omega^n)$, where $f^n \in H^{-1}(\Omega^n)$. The operator A is defined as

$$Au^n := \alpha\Delta u^n - \mathrm{div}(u^n w^n). \tag{3.9}$$

Hence, the variational formulation is : Suppose $u^0 \in H^1(\Omega^0)$, find $u^n \in H^1(\Omega^n)$, s.t. $\forall v \in H^1(\Omega^n)$ there holds

$$\int_{\Omega^n} \frac{(u^n - \mathcal{E}_{n-1} u^{n-1})}{\Delta t} v dx + a^n(u^n, v) = \int_{\Omega^n} f^n v dx, \quad n{=}1, 2, ..., N \tag{3.10}$$

where $a^n(\cdot,\cdot)$ defines the bilinear form for diffusion and convection parts.

Remark 9. *The coercivity of the bilinear form on the l.h.s of (3.10) implies that we can choose a suitable small time step and in each time step there is an unique solution. This is argued by Lemma 3.1 in [1].*

Remark 10. *In the spatial discretization the extension operator $\mathcal{E}_{n-1}$ from* (3.10) *is directly incorporated in the (stabilized) solution of the previous time step so that only one linear problem has to be solved in each time step of the implicit Euler method. The result of each linear solution step corresponds to $\mathcal{E}_n u^n$ so that it is defined already on Ω^{n+1}.*

3.2.2 Stabilized time stepping method based on BDF methods

In this section, we extend the implicit Euler time stepping method to BDF2 and BDF3 time stepping methods for moving domains.

Stabilized time stepping based on BDF2 method

We obtain the formulation by induction. In order to compute the approximated solution $u^2 \in H^1(\Omega^2)$ we need first to obtain the approximated solution $u^1 \in H^1(\Omega^1)$. Suppose $u^0 \in H^1(\Omega^0)$, we extend Ω^0 to its 2δ-neighborhood $\mathcal{O}_{2\delta}(\Omega^0)$ by an extension operator $\mathcal{E}_0 : H^1(\Omega^0) \to H^1(\mathcal{O}_{2\delta}(\Omega^0))$ s.t. $\Omega^1, \Omega^2 \subset \mathcal{O}_{2\delta}(\Omega^0)$. This setup makes u^0 and u^1 well-defined in {Ω^1, Ω^2}. When n=1, it is an

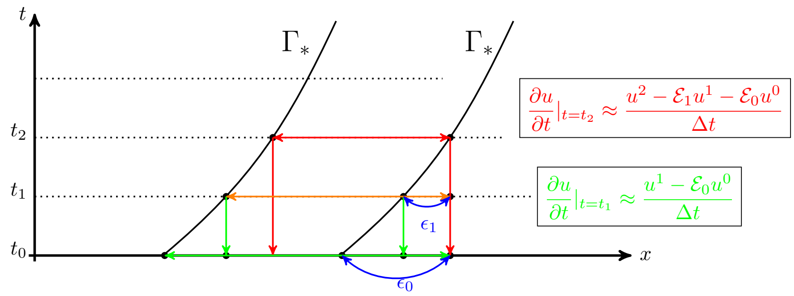

Figure 3.1: BDF2 method for moving domains. We extend Ω^0 to $\mathcal{O}_{2\delta}(\Omega^0)$ s.t. $\Omega^1 \subset \mathcal{O}_{2\delta}(\Omega^0)$. The green line illustrates an implicit step. Within the step we apply an extension operator $\mathcal{E}_0$ to u^0 to compute u^1. The orange line illustrates the extension from Ω^1 to $\mathcal{O}_\delta(\Omega^1)$. This setup makes $\Omega^2 \subset \mathcal{O}_\delta(\Omega^1) \subset \mathcal{O}_{2\delta}(\Omega^0)$. The red line illustrates the first BDF2 step. $\epsilon_0 := \mathcal{O}_{2\delta}(\Omega^0)\backslash\Omega^0$. $\epsilon_1 := \mathcal{O}_\delta(\Omega^1)\backslash\Omega^1$.

implicit Euler time stepping, we compute the approximated solution u^1 by using (3.8)

$$\frac{u^1 - \mathcal{E}_0 u^0}{\Delta t} = Au^1 + f^1, \tag{3.11}$$

where $Au^1 \in H^{-1}(\Omega^1)$ and $f^1 \in H^{-1}(\Omega^1)$.

When n=2, it is an BDF2 time stepping, we extend Ω^1 to its δ-neighborhood $\mathcal{O}_\delta(\Omega^1)$ by an extension operator $\mathcal{E}_1 : H^1(\Omega^1) \to H^1(\mathcal{O}_\delta(\Omega^1))$ s.t. u^1 is well-defined in Ω^2, then we obtain the approximated solution $u^2 \in H^1(\Omega^2)$ by

$$\frac{3}{2}u^2 - 2\mathcal{E}_1 u^1 + \frac{1}{2}\mathcal{E}_0 u^0 = \Delta t(Au^2 + f^2), \tag{3.12}$$

where $Au^2 \in H^{-1}(\Omega^2)$ and $f^2 \in H^{-1}(\Omega^2)$, cf. Figure 3.1.

By the analogy, we obtain the semi-discretization based on stabilized BDF2 time stepping method as : Suppose $u^0 \in H^1(\Omega^0)$, $u^1 \in H^1(\Omega^1)$, for $n = 2, 3, ..., N$ there holds

$$\frac{3u^n - 4\mathcal{E}_{n-1}u^{n-1} + \mathcal{E}_{n-2}u^{n-2}}{2\Delta t} = Au^n + f^n, \tag{3.13}$$

with $\mathcal{E}_{n-1} : H^1(\Omega^{n-1}) \to H^1(\mathcal{O}_\delta(\Omega^{n-1}))$, $\mathcal{E}_{n-2} : H^1(\Omega^{n-2}) \to H^1(\mathcal{O}_{2\delta}(\Omega^{n-2}))$, where $Au^n \in H^{-1}(\Omega^n)$ and $f^n \in H^{-1}(\Omega^n)$.

The variational form is : Suppose $u^0 \in H^1(\Omega^0)$, $u^1 \in H^1(\Omega^1)$, find $u^n \in H^1(\Omega^n)$, s.t. $\forall v \in H^1(\Omega^n)$, $n = 2, 3, ..., N$, there holds

$$\int_{\Omega^n} (\frac{3}{2}u^n - 2\mathcal{E}_{n-1}u^{n-1} + \frac{1}{2}\mathcal{E}_{n-2}u^{n-2})v dx + a^n(u^n, v) = \int_{\Omega^n} f^n v dx. \tag{3.14}$$

where $a^n(u^n, v)$ is the bilinear form for diffusion and convection parts.

Stabilized time stepping based on BDF3 method

Based on the stabilized implicit Euler and stabilized BDF2 semi-discretizations, we can obtain the semi-discretization based on stabilized BDF3 time stepping method as

$$\frac{11u^n - 18\mathcal{E}_{n-1}u^{n-1} + 9\mathcal{E}_{n-2}u^{n-2} - 2\mathcal{E}_{n-3}u^{n-3}}{6\Delta t} = Au^n + f^n, \quad n{=}3, 4, ..., N. \tag{3.15}$$

where $Au^n \in H^{-1}(\Omega^n)$ and $f^n \in H^{-1}(\Omega^n)$. The extension operator $\mathcal{E}_{n-3} : H^1(\Omega^{n-3}) \to H^1(\mathcal{O}_{3\delta}(\Omega^{n-3}))$ extends the functions on Ω^0 to the functions on $\mathcal{O}_{3\delta}(\Omega^0)$ s.t. $u^{n-3}, u^{n-2}, u^{n-1}$ are well-defined on Ω^n.

The variational form is :

Suppose $u^0 \in H^1(\Omega^0)$, $u^1 \in H^1(\Omega^1)$ and $u^2 \in H^1(\Omega^2)$, find $u^n \in H^1(\Omega^n)$, $\forall v \in H^1(\Omega^n)$, $n = 3, 4, ..., N$ there holds

$$\int_{\Omega^n} (\frac{11}{6}u^n - 3\mathcal{E}_{n-1}u^{n-1} + \frac{3}{2}\mathcal{E}_{n-2}u^{n-2} - \frac{1}{3}\mathcal{E}_{n-3}u^{n-3})vdx + a^n(u^n, v) = \int_{\Omega^n} f^n vdx, \tag{3.16}$$

where $a^n(u^n, v)$ is the bilinear form for diffusion and convection parts.

3.3 Stabilized time stepping based on explicit Euler

Compared to the implicit time stepping methods, the explicit time stepping methods calculate the new approximated solution based only on previous time levels.

3.3.1 The explicit time stepping method

A naive idea is that applying the extension first and then stepping on the new domain, i.e. we extend functions on Ω^0 to $\mathcal{O}_\delta(\Omega^0)$ by an extension operator $\mathcal{E}_0$ s.t. $\Omega^1 \subset \mathcal{O}_\delta(\Omega^0)$ and u^0 is well-defined in Ω^1. Suppose u^0 is well-defined on Ω^0, using the explicit Euler time stepping method we *formally* have

$$\frac{u^1 - \mathcal{E}_0 u^0}{\Delta t} = A\mathcal{E}_0 u^0 + \mathcal{E}_0 f^0 \quad \text{in } \Omega^1. \tag{3.17}$$

Remark 11. *The problem* (3.17) *is neither well-posed in $L^2(\Omega^1)$ nor in $H^1(\Omega^1)$. On the one hand the operator norm $\|A\| = sup\frac{<Av,v>}{\|v\|}$ is unbounded if the problem is posed in $L^2(\Omega^1)$. On the other hand the l.h.s. bilinear form is not coercive (and not inf-sup stable) if the problem is posed in $H^1(\Omega^1)$. This is why we treat the development of discretizations in this section only* formally. *The resulting discretizations are of value anyway as after spatial discretization the problem will become well-posed again. The operator norm $\|A\| = sup\frac{<Av,v>}{\|v\|}$ of the* discrete problem *will be bounded (depending on h though) even if the problem is posed only in $L^2(\Omega^1)$ (due to inverse inequalities).*

We assume now that u^1 is well-defined on Ω^1. Similarly, we extend functions on Ω^1 to $\mathcal{O}_\delta(\Omega^1)$ by

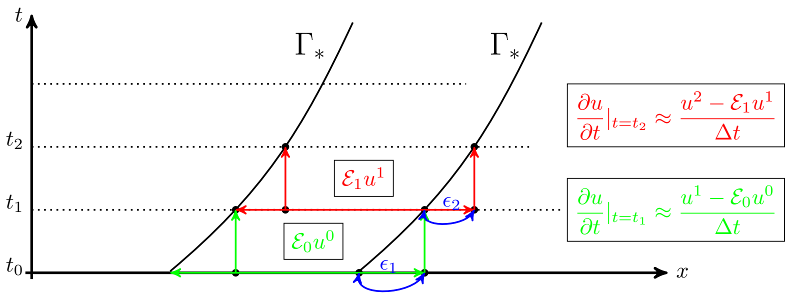

Figure 3.2: The naive explicit Euler method with an extension operator $\mathcal{E}$. The green line illustrates the first explicit step. We extend Ω^0 to $\mathcal{O}_\delta(\Omega^0)$ by using $\mathcal{E}_0$ s.t. Ω^1 is a subset of $\mathcal{O}_\delta(\Omega^0)$. $\epsilon_1 := \mathcal{O}_\delta(\Omega^0)\backslash\Omega^0$. The red line illustrates the second explicit step. We extend Ω^1 to $\mathcal{O}_\delta(\Omega^1)$ by using $\mathcal{E}_1$ s.t. Ω^2 is a subset of $\mathcal{O}_\delta(\Omega^1)$. $\epsilon_2 := \mathcal{O}_\delta(\Omega^1)\backslash\Omega^1$

an extension operator $\mathcal{E}_1$ s.t. $\Omega^2 \subset \mathcal{O}_\delta(\Omega^1)$ and u^1 is well-defined in Ω^2. Using the explicit Euler time stepping method we obtain (*formally*)

$$\frac{u^2 - \mathcal{E}_1 u^1}{\Delta t} = (A\mathcal{E}_1 u^1 + \mathcal{E}_1 f^1) \quad \text{in } \Omega^2, \tag{3.18}$$

cf. Figure (3.2).

By the analogy, we obtain the naive explict Euler time stepping method for moving domain problems : Suppose u^0 is well-defined on Ω^0 so that for n=1, 2, ..., N, there holds

$$\begin{aligned} u^1 &= \mathcal{E}_0 u^0 + \Delta t(A\mathcal{E}_0 u^0 + \mathcal{E}_0 f^0) \quad \text{in } \Omega^1, \\ u^2 &= \mathcal{E}_1 u^1 + \Delta t(A\mathcal{E}_1 u^1 + \mathcal{E}_1 f^1) \quad \text{in } \Omega^2, \\ &\vdots \\ u^n &= \mathcal{E}_{n-1} u^{n-1} + \Delta t(A\mathcal{E}_{n-1} u^{n-1} + \mathcal{E}_{n-1} f^{n-1}) \quad \text{in } \Omega^n. \end{aligned} \tag{3.19}$$

Another idea is that applying the explicit Euler time stepping method on the last domain first before extending to the current domain. Suppose u^0 is well-defined on Ω^0, then using the explicit Euler time stepping method we formally have

$$\frac{\tilde{u}^1 - u^0}{\Delta t} = Au^0 + f^0 \quad \text{in } \Omega^0. \tag{3.20}$$

Let us stress that the setting and conclusions from *Remark* 11 are also valid here. We assume $\tilde{u}^1$ is well-defined on Ω^0 and extend $\tilde{u}^1$ to u^1 by a proper extension $\mathcal{E}_0$ s.t. u^1 is well-defined in Ω^1. We obtain the approximated solution u^1=$\mathcal{E}_0\tilde{u}^1$. Similarly, using the explicit Euler time stepping method we have

$$\frac{\tilde{u}^2 - u^1}{\Delta t} = Au^1 + f^1 \quad \text{in } \Omega^1. \tag{3.21}$$

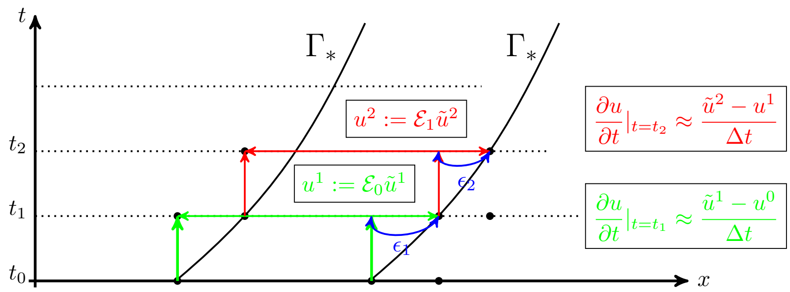

Figure 3.3: The explicit Euler method with an extension operator $\mathcal{E}$. The green line illustrates the first explicit step. We extend Ω^0 to $\mathcal{O}_\delta(\Omega^0)$ by using $\mathcal{E}_0$ s.t. Ω^1 is a subset of $\mathcal{O}_\delta(\Omega^0)$. $\epsilon_1 := \mathcal{O}_\delta(\Omega^0)\backslash\Omega^0$. The red line illustrates the second explicit step. We extend Ω^1 to $\mathcal{O}_\delta(\Omega^1)$ by using $\mathcal{E}_1$ s.t. Ω^2 is a subset of $\mathcal{O}_\delta(\Omega^1)$. $\epsilon_2 := \mathcal{O}_\delta(\Omega^1)\backslash\Omega^1$

Assume $\tilde{u}^2$ is well-defined in Ω^1 and extend $\tilde{u}^2$ to u^2 by $\mathcal{E}_1$ s.t. $\Omega^2 \in \mathcal{O}_\delta(\Omega^1)$ and u^2 is well-defined in Ω^2. We obtain the approximated solution u^2=$\mathcal{E}_1\tilde{u}^2$, cf. Figure (3.3).

By the analogy, we conclude an explicit Euler formulation for moving domains which different to the previous formulation in (3.19): Suppose u^0 is well-defined on Ω^0 so that for n=1, 2, ..., N, there holds

$$\begin{aligned} \tilde{u}^n &= u^{n-1} + \Delta t(Au^{n-1} + f^{n-1}) \quad \text{in } \Omega^{n-1}, \\ u^n &= \mathcal{E}_{n-1}\tilde{u}^n. \end{aligned} \tag{3.22}$$

We write down a *formal* variational formulation for each time step that becomes: Suppose u^{n-1} is given in $H^1(\Omega^{n-1})$ with sufficient regularity to evaluate $a^{n-1}(u^{n-1}, \cdot)$, find $\tilde{u}^n \in H^1(\Omega^{n-1})$ and $u^n \in H^1(\Omega^n)$, s.t. $\forall v \in H^1(\Omega^{n-1})$ and $w \in H^1(\Omega^n)$, $n = 1, 2, ..., N$, there holds

$$\int_{\Omega^{n-1}} \frac{\tilde{u}^n - u^{n-1}}{\Delta t} v dx + a^{n-1}(u^{n-1}, v) = \int_{\Omega^{n-1}} f^{n-1} v dx. \tag{3.23a}$$

$$\int_{\Omega^n} u^n w dx = \int_{\Omega^n} \mathcal{E}_{n-1}\tilde{u}^n w dx. \tag{3.23b}$$

where $a^{n-1}(u^{n-1}, v)$ defines the bilinear form for diffusion and convection corresponding to Ω^{n-1}. We note that according to Remark 11 the variational formulation is not well-posed. However, after spatial discretization it will become well-posed due to the equivalence of discrete version of the H^1 and L^2 spaces (with constants depending on h) if proper stabilizations are applied (to deal with cut elements).

Remark 12. *In the spatial discretization the extension operator $\mathcal{E}_{n-1}$ from* (3.23b) *is incorporated already in the (stabilized) solution in* (3.23a) *so that only one linear problem has to be solved in each time step of the explicit Euler method.*

3.4 Trapezoidal rule (TR) for moving domains

The *trapezoidal rule* is an implicit second order numerical method to solve ODEs which is also considered as both a Runge-Kutta method and a linear multistep method. For example, consider the *initial value problem*

$$u' = F(t,u),\ u(0) = u_0 \tag{3.24}$$

The trapezoidal rule is given by

$$u^n - u^{n-1} = \frac{1}{2}\Delta t(F(t_n,u^n) + F(t_{n-1},u^{n-1})),\ n{=}1,2,...N \tag{3.25}$$

with time step Δt. One possible method for solving (3.25) is the Newton's method if F is nonlinear.

3.4.1 Different characterizations of TR

Below, we want to generalize the trapezoidal rule to moving domains. To this end, we first give three different characterizations of the standard method:

1. We can rewrite the trapezoidal rule as a subsequent execution of half time step of explicit Euler and half time step of an implicit Euler method yielding the following characterization

$$\begin{cases} u^{n+\frac{1}{2}} = u^n + \frac{\Delta t}{2}F(t_n,u^n) \\ u^{n+1} = u^{n+\frac{1}{2}} + \frac{\Delta t}{2}F(t_{n+1},u^{n+1}) \end{cases}$$

2. Alternatively, in the linear case $F(u^n,t_n) = Au^n + f^n$ we can characterize the trapezoidal rule as the average of an explicit and an implicit Euler step as can be seen from the following characterization

$$\begin{cases} u_e^{n+1} = u^n + \Delta t(Au^n + f^n) \\ u_i^{n+1} = u^n + \Delta t(Au^{n+1} + f^{n+1}) \\ u^{n+1} = \frac{1}{2}(u_e^{n+1} + u_i^{n+1}) \end{cases}$$

3. As a last characterization we directly consider (3.25):

$$u^n - u^{n-1} = \frac{1}{2}\Delta t(F(t_n,u^n) + F(t_{n-1},u^{n-1})),\ n{=}1,2,...N \tag{3.25}$$

3.4.2 Trapezoidal rules for moving domains

Based on the previously mentioned characterizations, we construct different generalizations which can in principle be applied to moving domains.

A trapezoidal rule based on subsequent Euler half steps

We divide the global time step Δt into two uniform sized substeps $\frac{\Delta t}{2}$. In the first substep is an explicit Euler time step, in the second substep is an implicit Euler time step. As we have discussed in Section 3.3 there are two strategies for the explicit Euler time stepping. One is applying the extension first and then stepping on the new domain, while the other does the time stepping first and then extension. Here we use latter.

Suppose u^0 is well-defined on Ω^0, applying (3.22) we have

$$\tilde{u}^{\frac{1}{2}} - u^0 = \frac{\Delta t}{2}(Au^0 + f^0), \quad \text{in } \Omega^0 \tag{3.26}$$

we refer to Remark 11 for the (non-)well-poseness of this problem on the continuous and disrete level. We extend Ω^0 to its $\frac{\delta}{2}$-neighborhood $\mathcal{O}_{\frac{\delta}{2}}(\Omega^0)$ by an extension operator $\mathcal{E}_{0,\frac{1}{2}}$ s.t. $\Omega^{\frac{1}{2}} \subset \mathcal{O}_{\frac{\delta}{2}}(\Omega^0)$ and $u^{\frac{1}{2}}$ is well-defined in $\Omega^{\frac{1}{2}}$, we obtain the appoximated solution by

$$u^{\frac{1}{2}} := \mathcal{E}_{0,\frac{1}{2}}\tilde{u}^{\frac{1}{2}}, \quad \text{in } \Omega^{\frac{1}{2}} \subset \mathcal{O}_{\frac{\delta}{2}}(\Omega^0). \tag{3.27}$$

Next, we extend $\Omega^{\frac{1}{2}}$ to its $\frac{\delta}{2}$-neighborhood $\mathcal{O}_{\frac{\delta}{2}}(\Omega^{\frac{1}{2}})$ by an extension operator $\mathcal{E}_{\frac{1}{2},\frac{1}{2}}$ s.t. $\Omega^1 \subset \mathcal{O}_{\frac{\delta}{2}}(\Omega^{\frac{1}{2}})$ and u^1 is well defined in Ω^1. Then applying (3.8) we obtain

$$u^1 - \mathcal{E}_{\frac{1}{2},\frac{1}{2}}u^{\frac{1}{2}} = \frac{1}{2}\Delta t(Au^1 + f^1), \quad \text{in } \Omega^1. \tag{3.28}$$

Substituting (3.26) into (3.27) and combining with (3.28) we obtain

$$u^1 - \mathcal{E}_1 u^0 = \frac{\Delta t}{2}(Au^1 + f^1 + \mathcal{E}_1(Au^0 + f^0)), \tag{3.29}$$

where $\mathcal{E}_1 := \mathcal{E}_{\frac{1}{2},\frac{1}{2}}\mathcal{E}_{0,\frac{1}{2}}$, cf. Figure (3.4) for an example. By the analogy, we obtain the first characterization of trapezoidal rule for moving domains as :

Suppose u^0 is well-defined on Ω^0, find u^n in Ω^n, for $n = 1, 2, ..., N$ there holds

$$u^n - \mathcal{E}_n u^{n-1} = \frac{1}{2}\Delta t(Au^n + f^n + \mathcal{E}_n(Au^{n-1} + f^{n-1})), \tag{3.30}$$

where $\mathcal{E}_n := \mathcal{E}_{\frac{2n-1}{2},\frac{2n-1}{2}}\mathcal{E}_{n-1,\frac{2n-1}{2}}$. The variational form is given by:

Suppose $u^0 \in H^1(\Omega^0)$, find $u^n \in H^1(\Omega^n)$ s.t. for $\forall v \in H^1(\Omega^n)$, n=$1, 2, ..., N$ there holds

$$\int_{\Omega^n} \frac{u^n - \mathcal{E}_n u^{n-1}}{\frac{\Delta t}{2}} v dx + a^n(u^n, v) + \mathcal{E}_n a^{n-1}(u^{n-1}, v) = \int_{\Omega^n} f^n v dx + \mathcal{E}_n f^{n-1} v dx \tag{3.31}$$

where $a^n(u^n, v)$ and $a^{n-1}(u^{n-1}, v)$ are the bilinear forms for diffusion and convection parts.

Remark 13. *The problem (3.31) is well-poseness as $\mathcal{E}_n$ is a bounded linear operator in H^{-1}, cf. Lemma 1.*

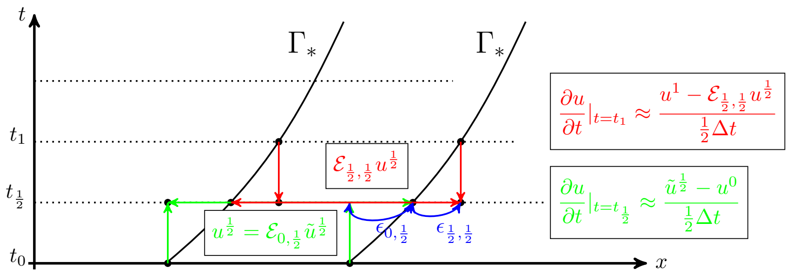

Figure 3.4: The first characterization. The green line represents an explicit Euler step. We extend Ω^0 to $\mathcal{O}_{\frac{\delta}{2}}(\Omega^0)$ by an extension operator $\mathcal{E}_{0,\frac{1}{2}}$ s.t. $\Omega^{\frac{1}{2}} \subset \mathcal{O}_{\frac{\delta}{2}}(\Omega^0)$. $\epsilon_{0,\frac{1}{2}} := \mathcal{O}_{\frac{\delta}{2}}(\Omega^0)\backslash\Omega^0$. The red line represents an implicit Euler step. We extend $\Omega^{\frac{1}{2}}$ to $\mathcal{O}_{\frac{\delta}{2}}(\Omega^{\frac{1}{2}})$ by using an extension operator $\mathcal{E}_{\frac{1}{2},\frac{1}{2}}$ s.t. $\Omega^1 \subset \mathcal{O}_{\frac{\delta}{2}}(\Omega^{\frac{1}{2}})$. $\epsilon_{\frac{1}{2},\frac{1}{2}} := \mathcal{O}_{\frac{\delta}{2}}(\Omega^{\frac{1}{2}})\backslash\Omega^{\frac{1}{2}}$

.

A trapezoidal rule based on averaged Euler steps

Second characterization is an average of the implicit and the modified explicit Euler steps.

Suppose u^0 is well-defined on Ω^0, applying (3.22) we have

$$\frac{\tilde{u}^1 - u^0}{\Delta t} = Au^0 + f^0, \quad \text{in } \Omega^0. \tag{3.32}$$

we refer to Remark 11 for the (non-)well-poseness of this problem on the continuous and disrete level. We extend Ω^0 to $\mathcal{O}_\delta(\Omega^0)$ by a proper extension operator $\mathcal{E}_0$ s.t. $\Omega^1 \subset \mathcal{O}_\delta(\Omega^0)$ and u^1 is well-defined in Ω^1. We obtain the approximated solution u^1 by

$$u^1 := \mathcal{E}_0\tilde{u}^1, \quad \text{in } \Omega^1 \tag{3.33}$$

and we set $u_e^1 := u^1$. Similarly, applying (3.8) we obtain

$$u^1 - \mathcal{E}_0 u^0 = \Delta t(Au^1 + f^1), \quad \text{in } \Omega^1 \tag{3.34}$$

and we set $u_i^1 := u^1$, cf. Figure (3.5) for an example.

Hence, we obtain the approximated solution u^1 by the computation of average value of u_e^1 and u_i^1

$$\begin{aligned} u^1 &= \frac{u_e^1 + u_i^1}{2} \\ &= \frac{1}{2}(\mathcal{E}_0 u^0 + \Delta t\mathcal{E}_0(Au^0 + f^0)) + \frac{1}{2}(\mathcal{E}_0 u^0 + \Delta t(Au^1 + f^1)) \\ &= \mathcal{E}_0 u^0 + \frac{1}{2}\Delta t((Au^1 + f^1) + \mathcal{E}_0(Au^0 + f^0)). \end{aligned} \tag{3.35}$$

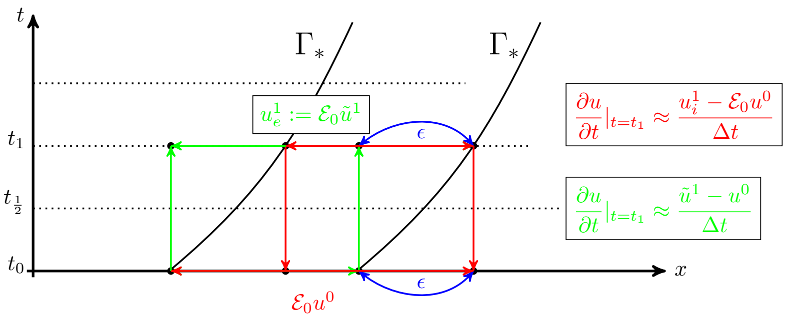

Figure 3.5: The second characterization. The green line illustrates that we compute $\tilde{u}^1$ in an explicit step and then extend it to $u_e^1 : \mathcal{E}_0\tilde{u}^1$. The red line illustrates an implicit step within which we compute u_i^1. $\epsilon := \mathcal{O}_\delta(\Omega^0)\backslash\Omega^0$.

By the analogy, the second characterization of the trapezoidal rule for moving domains is given by

$$u^n - \mathcal{E}_{n-1}u^{n-1} = \frac{1}{2}\Delta t(Au^n + f^n + \mathcal{E}_{n-1}(Au^{n-1} + f^{n-1})),\ n{=}1, 2, ..., N \tag{3.36}$$

The variational form is given by :

Suppose u^0 is well-defined on Ω^0, find u^n on Ω^n so that for $\forall v \in H^1(\Omega^n),\ n{=}1, 2, ..., N$ there holds

$$\int_{\Omega^n} \frac{u^n - \mathcal{E}_{n-1}u^{n-1}}{\frac{\Delta t}{2}} v dx + a^n(u^n, v) + a^{n-1}(\mathcal{E}_{n-1}u^{n-1}, v) = \int_{\Omega^n} f^n v dx + \mathcal{E}_{n-1}f^{n-1}v dx \tag{3.37}$$

where $a^n(u^n, v)$ and $a^{n-1}(\mathcal{E}_{n-1}u^{n-1}, v)$ are the bilinear forms for diffusion and convection parts.
Remark 14. *The difference between the extension operator in (3.31) and (3.37) is that $\mathcal{E}_n$ in (3.31) is a two-steps extenstion operator while $\mathcal{E}_{n-1}$ in (3.37) is a one-step extension operator.*

A naive trapezoidal rule for moving domains

Relative to first and second characterizations, we obtain the last characterization directly by substituting u to its extension $\mathcal{E}u$ and rewrite the formular (3.25) as

$$u^n - \mathcal{E}_{n-1}u^{n-1} = \frac{1}{2}\Delta t(Au^n + f^n + (A\mathcal{E}_{n-1}u^{n-1} + f^{n-1})), \quad \text{in } \Omega^n. \tag{3.38}$$

The variational form is given by:

Suppose u^0 is well-defined on Ω^0, find u^n in Ω^n so that for $\forall v \in H^1(\Omega^n),\ n{=}1, 2, ..., N$ there holds

$$\int_{\Omega^n} \frac{u^n - \mathcal{E}_{n-1}u^{n-1}}{\frac{\Delta t}{2}} v dx + a^n(u^n, v) + a^{n-1}(\mathcal{E}_{n-1}u^{n-1}, v) = \int_{\Omega^n} f^n v dx + f^{n-1}v dx \tag{3.39}$$

where $a^n(u^n, v)$ and $a^{n-1}(\mathcal{E}_{n-1}u^{n-1}, v)$ are the bilinear forms for diffusion and convection parts.

Observation

We observe that for $\mathcal{E}_{n-1}A = A\mathcal{E}_{n-1}$ the previously discussed formulations (3.36) and (3.38) coincide. However, we can not expect $A\mathcal{E}_{n-1} = \mathcal{E}_{n-1}A$ to hold in general.

3.5 Midpoint rule for moving domains

The discussion on midpoint rule for moving domains is similar to the discussion on first characterization of trapezoidal rule for moving domains. We divide global time step Δt into two uniform sized substeps $\frac{\Delta t}{2}$. In the first substep is an implicit Euler time step, in the second substep is an explicit Euler time step. As we have discussed in Section 3.3 there are two strategies for the explicit Euler time stepping. One is applying the extension first and then stepping on the new domain, while the other does the time stepping first and then extension. Here we use latter.

We extend Ω^0 to its $\frac{\delta}{2}$-neighborhood $\mathcal{O}_{\frac{\delta}{2}}$ by an extension operator $\mathcal{E}_0$ s.t. $\Omega^{\frac{1}{2}} \subset \mathcal{O}_{\frac{\delta}{2}}(\Omega^0)$ and $u^{\frac{1}{2}}$ is well-defined in $\Omega^{\frac{1}{2}}$. Applying (3.8) we have

$$u^{\frac{1}{2}} - \mathcal{E}_0 u^0 = \frac{\Delta t}{2}(Au^{\frac{1}{2}} + f^{\frac{1}{2}}), \tag{3.40}$$

where $Au^{\frac{1}{2}} \in H^{-1}(\Omega^{\frac{1}{2}})$ and $f^{\frac{1}{2}} \in H^{-1}(\Omega^{\frac{1}{2}})$.

Next, applying (3.22) we have

$$\tilde{u}^1 - u^{\frac{1}{2}} = \frac{\Delta t}{2}(Au^{\frac{1}{2}} + f^{\frac{1}{2}}), \quad \text{in}\Omega^{\frac{1}{2}}. \tag{3.41}$$

As we have discussed above, we refer to Remark 11 for the (non-)well-poseness of this problem on the continuous and discrete level. Substituting (3.40) into (3.41) we obtain

$$\tilde{u}^1 - \mathcal{E}_0 u^0 = \Delta t(Au^{\frac{1}{2}} + f^{\frac{1}{2}}). \tag{3.42}$$

We extend $\Omega^{\frac{1}{2}}$ to its $\frac{\delta}{2}$-neighborhood $\mathcal{O}_{\frac{\delta}{2}}(\Omega^{\frac{1}{2}})$ by an extension operator $\mathcal{E}_{\frac{1}{2}}$ s.t. Ω^1 is a subset of $\mathcal{O}_{\frac{\delta}{2}}(\Omega^{\frac{1}{2}})$ and u^1 is well-defined in Ω^1. We obtain the approximated solution by

$$u^1 := \mathcal{E}_{\frac{1}{2}}\tilde{u}^1. \tag{3.43}$$

Hence, by the analogy we obtain the midpoint rule for moving domains : Suppose $u^0 \in H^1(\Omega^0)$, find $u^n \in H^1(\Omega^n)$, for $n = 1, 2, ..., N$ there holds

$$\begin{aligned} u^{n'} &= \mathcal{E}_{n-1}u^{n-1} + \frac{1}{2}\Delta t(Au^{n'} + f^{n'}), \\ \tilde{u}^n &= u^{n'} + \Delta t(Au^{n'} + f^{n'}), \\ u^n &= \mathcal{E}_{n'}\mathcal{E}_{n-1}u^{n-1} + \Delta t\mathcal{E}_{n'}(Au^{n'} + f^{n'}), \end{aligned} \tag{3.44}$$

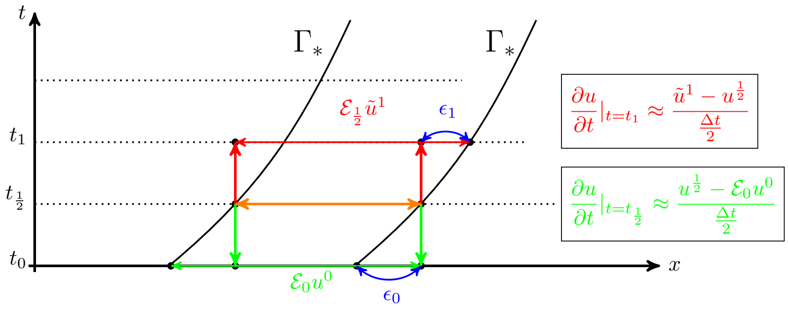

Figure 3.6: Midpoint rule for moving domains. We extend Ω^0 to $\mathcal{O}_{\frac{\delta}{2}}(\Omega^0)$ s.t. $\Omega^{\frac{1}{2}} \subset \mathcal{O}_{\frac{\delta}{2}}(\Omega^0)$ and $u^{\frac{1}{2}}$ is well-defined in $\Omega^{\frac{1}{2}}$. The green line represents an implicit step, we obtain the approximated solution $u^{\frac{1}{2}}$. $\epsilon_0 := \mathcal{O}_{\frac{\delta}{2}}(\Omega^0) \backslash \Omega^0$. The red line represents an explicit step. Within the explicit step we compute $\tilde{u}^1$. We assume $\tilde{u}^1 \in H^1(\Omega^1)$, apply an extension operator $\mathcal{E}_{\frac{1}{2}}$ to $\tilde{u}^1$ to obtain u^1. $\epsilon_1 := \mathcal{O}_{\frac{\delta}{2}}(\Omega^{\frac{1}{2}}) \backslash \Omega^{\frac{1}{2}}$

where $n' := \frac{2n-1}{2}$.

The variational form is given as :

Suppose $u^0 \in H^1(\Omega^0)$, find $u^n \in H^1(\Omega^n)$ so that for $\forall v \in H^1(\Omega^n)$, n=1, 2, ..., N there holds

$$\int_{\Omega^n} \frac{u^n - \mathcal{E}_{n'}\mathcal{E}_{n-1}u^{n-1}}{\Delta t} v dx + \mathcal{E}_{n'} a^{n'}(u^{n'}, v) = \int_{\Omega^n} \mathcal{E}_{n'} f^{n'} v dx \tag{3.45}$$

where $a^{n'}(u^{n'}, v)$ is the bilinear form for difussion and convection parts.

3.6 Summary

In this chapter, we discuss several higher oder stabilized time stepping methods for moving domains. In these methods, the solution for u^n on the domain Ω^n is combined with its extension on $\mathcal{O}_\delta(\Omega^n)$ in each time step. In the next chapter, the extension can be realized by a stabilization term in the numerical methods.

Chapter 4

Full discretization method

In chapter 3, based on first order implicit Euler and explicit Euler time stepping methods for moving domains, we have discussed several higher order time stepping methods. In each time step we extend domain Ω^n by a layer of thinkness δ s.t. Ω^{n+1} is a subset of the extended domain to Ω^n, i.e. $\mathcal{O}_\delta(\Omega^n)$. This setup makes u^n well-defined in Ω^{n+1}. In this chapter, we realize the extension operator $\mathcal{E}_n$ by a stabilization term which is combined with the approximated solution in the full discretization mehtod.

4.1 Motivation

Unfitted FEM provides an extension of FEM to deal with such as strong deformation of geometry or topology changes. which relys on basic idea of choosing FEM based approximation of the physical fields independently. Cut finite element method (CutFEM) which is introduced in details in [2] is a numerical method for solving cutting off finite elements and their associated discrete approaches at boundaries and interfaces of the domains, cf. Figure 4.1 for an example.

Remark 15 (Ghost penalty)**.** *Cutting the mesh can lead to a problem that it leads boundary elements to very small intersections with physical domain. This may cause ill-conditioned system matrix or failure of stability of the numerical scheme. A useful mehtod which is introduced in [13] is to add a penalty term around the boundary that extends the coercivity to active mesh. This penalty term must be carefully designed to add sufficient stability, while remaining weakly consistent for smooth solutions.*

4.2 Full discretization for the implicit methods

Recall the unfitted finte element space $V_h^{n,l}$ refers to chapter 2 section 2.2.1

$$V_h^{n,l} := \{v \in C(\mathcal{O}_{\delta_h,\mathcal{T}}^{n,l}) : v \in \mathbb{P}_k(\mathcal{S}), \forall \mathcal{S} \in \mathcal{T}_h^{n,l}\}, \quad k \geq 1. \tag{4.1}$$

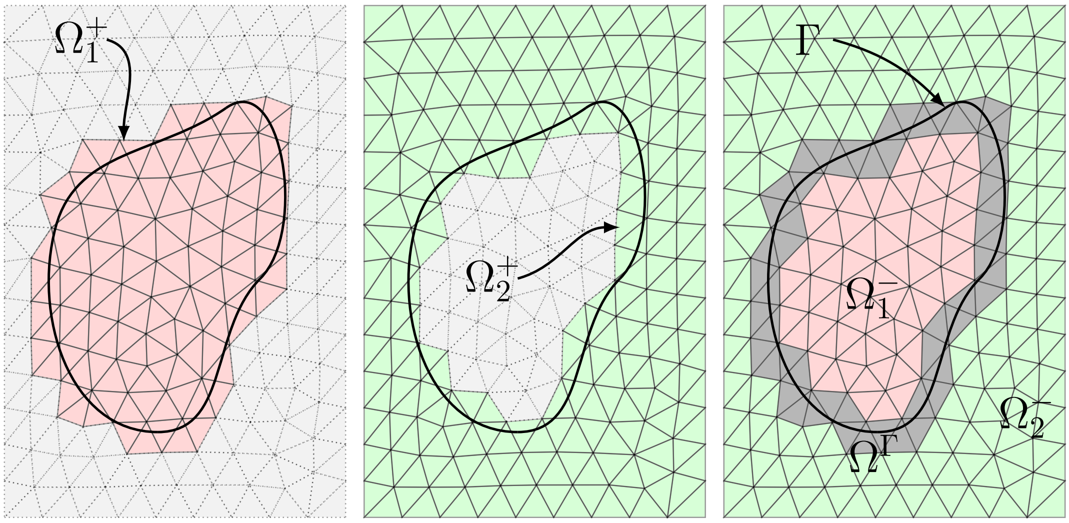

Figure 4.1: CutFEM. In left sketch, the pink mesh is active mesh which contains moving domain. In middel sketch, the white mesh is the mesh which is entirely contained in moving domain. In right sketch, the grey mesh represents the mesh cutted by the boundary of moving doamin. In the right sketch we can find that there are some elements cutted by the boundary with very small intersection parts, this may lead to a ill-conditioned system matrix, a penalty term will be added around the boundary to extend the coercivity to the active mesh.

where $\mathbb{P}_k(\mathcal{S})$ is the space of polynomials of at most degree k on $\mathcal{S}$. The elements which are in a boundary strip are defined as

$$\mathcal{T}_{\mathcal{S}^{\pm}}^{n,l} := \{\mathcal{S} \in \mathcal{T}_{\delta}^{n,l} : \text{dist}(\mathbf{x}, \Gamma_h^n) \leq l \cdot \delta_h \text{for some } \mathbf{x} \in \mathcal{S}\}. \tag{4.2}$$

Here, we recall the definition of $\mathcal{T}_{\delta}^{n,l}$, cf. chapter 2, where l indicates the number of time steps in which the domains $\Omega_h^n, \Omega_h^{n+1}, \ldots, \Omega_h^{n+l}$ are still contained in the domain corresponding to $\mathcal{T}_{\delta}^{n,l}$. In the boundary strip there are cutted elements and the elements completely inside or outside Ω_h^n. The set of facests between elements in $\mathcal{T}_{\delta}^{n,l}$ and $\mathcal{T}_{\mathcal{S}^{\pm}}^{n,l}$ is defined as

$$\mathcal{F}_h^{n,l} := \{\bar{T}_1 \bigcap \bar{T}_2 : T_1 \in \mathcal{T}_{\delta}^{n,l}, T_2 \in \mathcal{T}_{\mathcal{S}^{\pm}}^{n,l}, T_1 \neq T_2, meas_{d-1}(T_1 \bigcap T_2) > 0\}. \tag{4.3}$$

cf. Figure 4.2 for an example. If no index l is used, we assume $l = 1$, i.e. we define $\mathcal{T}_{\mathcal{S}^{\pm}}^{n} := \mathcal{T}_{\mathcal{S}^{\pm}}^{n,1}$, $\mathcal{F}_h^n := \mathcal{F}_h^{n,1}$.

4.2.1 Stabilization bilinear form

In [1], C.Lehrenfeld and M.A.Olshanskii introduce three possible choices of the stabilization term. In this thesis, we use one of these three stabilization terms : the "Direct" version of the ghost penaly stabilization. This version has the advantage that an implementation of the bilinear form is only implicitly through the extension $\mathcal{E}^{\mathbb{P}}$ depending on the polynomial degree k. It is proposed for the

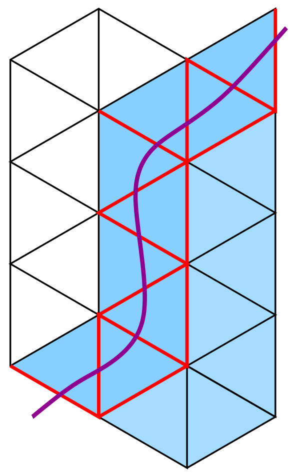

Figure 4.2: The purple line represents $\Gamma_h^n := \Omega_h^n$. Thick blue meshes are cutted elements cutted by Γ_h^n. Light blue meshes are elements completely outside Ω_h^n and white meshes are elements completely inside Ω_h^n The right line represents the facets in $\mathcal{F}_h^n$.

first time in [14].

Suppose $F \in \mathcal{F}_h^{n,l}$, w_F be the facet patch, i.e.

$$w_F = T_1 \bigcup T_2,\ T_1 \in \mathcal{T}_h^n,\ T_2 \in \mathcal{T}_{\mathcal{S}^\pm}^n. \tag{4.4}$$

Then for $u, v \in V_h^{n,l}$,

$$s_h^{n,l}(u, v) := \sum_{F \in \mathcal{F}_h^{n,l}} \frac{1}{h^2} \int_{w_F} (u_1 - u_2)(v_1 - v_2) dx, \tag{4.5}$$

where

$$u_1 = \mathcal{E}^{\mathbb{P}} u|_{T_1}, \quad u_2 = \mathcal{E}^{\mathbb{P}} u|_{T_2}, \tag{4.6}$$

and the operator

$$\mathcal{E}^{\mathbb{P}} : \mathbb{P}_k(S) \to \mathbb{P}_k(\mathbb{R}^d) \tag{4.7}$$

is the canonical extension operator of a polynomial to $\mathbb{R}^d$.

Remark 16. *The role of stabilization term $s_h^{n,l}(\cdot,\cdot)$ is responsible for the implicit definition of an extension to $\Omega_{\delta,\mathcal{T}}^{n,l}$. It provides condition number bounds that are independent of the cut position, cf. Remark 5.3 in [1] for details.*

Remark 17. *There are other variants of the ghost penalty stabilization in* (4.5)*. All of variants have in common that they add stabilization term on facets in the region of the boundary Γ_h^n, they also share the same important properties. We do not discuss those here, but instead refer to [1].*

4.2.2 Full discretization

Stabilized time stepping based on the implicit Euler method

The numerical method is based on (3.10). We extend Ω_h^n to $\mathcal{O}_{\delta_h,\mathcal{T}}^n$ s.t. $\Omega_h^n \subset \mathcal{O}_{\delta_h,\mathcal{T}}^n$. We realize the discrete extension by a stabilization term $s_h^n(\cdot,\cdot)$. The stabilization term $s_h^n(\cdot,\cdot)$ guarantees that u_h^n is well defined in Ω_h^n. Then, the full discretization read as :

Suppose $u_h^0 \in V_h^0$, find $u_h^n \in V_h^n$ so that for $n = 1, 2, ..., N$ there holds

$$\int_{\Omega_h^n} \frac{u_h^n - u_h^{n-1}}{\Delta t} v_h + a_h^n(u_h^n, v_h) + \gamma_s s_h^n(u_h^n, v_h) = \int_{\Omega_h^n} f_h^n v_h, \quad \forall v_h \in V_h^n, \tag{4.8}$$

where $\gamma_s = \gamma_s(h, \delta_h)$ is a stabilization parameter and bilinear form $a_h^n(u_h, v_h)$ is defined as

$$\begin{aligned} a_h^n(u_h, v_h) &:= \int_{\Omega_h^n} \nu \nabla u_h \cdot \nabla v_h dx + \frac{1}{2} \int_{\Omega_h^n} (\mathbf{w}^e \cdot \nabla u_h) v_h - (\mathbf{w}^e \cdot \nabla v_h) u_h dx \\ &+ \frac{1}{2} \int_{\Omega_h^n} \mathrm{div}(\mathbf{w}^e) u_h v_h dx + \frac{1}{2} \int_{\Gamma_h^n} (\mathbf{w}^e \cdot \mathbf{n}) u_h v_h dx, \ u_h, v_h \in H^1(\Omega_h^n), \end{aligned} \tag{4.9}$$

$\mathbf{w}^e$ is a suitable smooth extension of the velocity field $\mathbf{w}$ and it only depends on $\Omega^n \neq \Omega_h^n$.

Stabilized time stepping based on BDF2 method

The numerical method is based on (3.14). When n=1, it is an implicit Euler time stepping. The discrete extension is realized by a stabilization term $s_h^{1,2}(u_h^1, v_h)$ s.t. $\Omega_h^2, \Omega_h^3 \subset \mathcal{O}_{2\delta_h,\mathcal{T}}^1$ and $s_h^{1,2}(u_h^1, v_h)$ guarantees that u_h^1 is well defined in {Ω_h^2, Ω_h^3}. By (4.8) we have

$$\int_{\Omega_h^1} \frac{u_h^1 - u_h^0}{\Delta t} v_h + a_h^1(u_h^1, v_h) + \gamma_s s_h^{1,2}(u_h^1, v_h) = \int_{\Omega_h^1} f_h^1 v_h, \ \forall v_h \in V_h^1. \tag{4.10}$$

When n=2, it is a BDF2 time stepping. The discrete extension is realized by a stabilization term $s^{2,2}(u_h^2, v_h)$ s.t. $\Omega_h^4 \subset \mathcal{O}_{\delta_h,\mathcal{T}}^2$ and $s^{2,2}(u_h^2, v_h)$ guarantees that u_h^2 is well defined in {Ω_h^3,Ω_h^4}, then we have

$$\int_{\Omega_h^2} \frac{3u_h^2 - 4u_h^1 + u_h^0}{2\Delta t} v_h + a_h^2(u_h^2, v_h) + \gamma_s s_h^{2,2}(u_h^2, v_h) = \int_{\Omega_h^2} f_h^2 v_h, \ \forall v_h \in V_h^2. \tag{4.11}$$

Hence, by the analogy the full discretezation read as : Suppose $u^0 \in V_h^0$, $u^1 \in V_h^1$, find $u_h^n \in V_h^n$, for $n = 2, 3, ..., N$ there holds

$$\int_{\Omega_h^n} \frac{3u_h^n - 4u_h^{n-1} + u_h^{n-2}}{2\Delta t} v_h + a_h^n(u_h^n, v_h) + \gamma_s s_h^{n,2}(u_h^n, v_h) = \int_{\Omega_h^n} f_h^n v_h, \ \forall v_h \in V_h^n, \tag{4.12}$$

where $\gamma_s = \gamma_s(h, \delta_h)$ is a stabilization parameter and bilinear form $a_h^n(u_h, v_h)$ is defined as (4.9).

Stabilized time stepping based on BDF3 method

The numerical method is based on (3.16). Assume u_h^{n-1} well-defined in $\Omega_h^n,\ \Omega_h^{n+1},\ \Omega^{n+2}$, u^{n-2} well-defined on $\Omega_h^n,\ \Omega_h^{n+1}$ and u^{n-3} well-defined on Ω_h^n, then the full discretization read as :

Suppose $u_h^0 \in V_h^0$, $u_h^1 \in V_h^1$ and $u_h^2 \in V_h^2$, n=3, 4, ..., N satisfying

$$\int_{\Omega_h^n} \frac{11u_h^n - 18u_h^{n-1} + 9u_h^{n-2} - 2u_h^{n-3}}{6\Delta t} v_h + a_h^n(u_h^n, v_h) + \gamma_s s_h^{n,3}(u_h^n, v_h) = \int_{\Omega_h^n} f_h^n v_h, \quad \forall v_h \in V_h^n. \tag{4.13}$$

where $\gamma_s = \gamma_s(h, \delta_h)$ is a stabilization parameter and bilinear form $a_h^n(u_h, v_h)$ is defined as (4.9). $\mathbf{w}^e$ is a suitable smooth extension of the velocity field $\mathbf{w}$ and it only depends on $\Omega^n \neq \Omega_h^n$. The stabilization term $s^{n,3}(u_h^n, v_h)$ guarantees that $\Omega_h^{n+1}, \Omega_h^{n+2}, \Omega_h^{n+3} \subset \mathcal{O}_{3\delta_h,\mathcal{T}}^{n-3}$ and u_h^n is well defined in Ω_h^{n+3}, Ω_h^{n+2} and Ω_h^{n+1}.

4.3 Full discretization for the explicit method

The numerical method is based on (3.22). When n=1, the discrete extension is realized by a stabilization term $s_h^0(u_h^1, v_h)$ and $s_h^0(u_h^1, v_h)$ guarantees u^1 well-defined in Ω^1. Then the full discretization read as :

$$\int_{\Omega_h^0} \frac{u_h^1 - u_h^0}{\Delta t} v_h + a_h^0(u_h^0, v_h) + \gamma_s s_h^0(u_h^1, v_h) = \int_{\Omega_h^0} f_h^0 v_h, \quad \forall v_h \in V_h^0. \tag{4.14}$$

Hence, by the analogy we have: Suppose $u_h^0 \in V_h^0$, find $u_h^n \in V_h^n$, for n=1, 2, ..., N there holds

$$\int_{\Omega_h^{n-1}} \frac{u_h^n - u_h^{n-1}}{\Delta t} v_h + a_h^{n-1}(u_h^{n-1}, v_h) + \gamma_s s_h^{n-1}(u_h^n, v_h) = \int_{\Omega_h^{n-1}} f_h^{n-1} v_h, \quad \forall v_h \in V_h^{n-1}. \tag{4.15}$$

where $\gamma_s = \gamma_s(h, \delta_h)$ is a stabilization parameter and bilinear form $a_h^{n-1}(u_h, v_h)$ is defined as (4.9).

4.4 Full discretizations of trapezoidal rules

4.4.1 Full discretization of first characterization

As we have discussed in chapter 3, the first characterization of trapezoidal rule is trapezodial rule based on subsequent half Euler steps. The numerical method is based on (3.31). We divide global time step Δt into two uniform sized substeps $\frac{\Delta t}{2}$. In the first substep is an explicit Euler time step. We extend Ω_h^0 to its $\frac{\delta_h}{2}$-neighborhood $\mathcal{O}_{\frac{\delta_h}{2},\mathcal{T}}^0$. This discrete extension can be realized by a stabilization term $s_h^{0,\frac{1}{2}}(u_h^{\frac{1}{2}}, v_h)$ and $s^{0,\frac{1}{2}}(u_h^{\frac{1}{2}}, v_h)$ guarantees that $u_h^{\frac{1}{2}}$ is well defined in $\Omega_h^{\frac{1}{2}}$. Then suppose $u^0 \in V_h^0$ we have

$$\int_{\Omega_h^0} \frac{u_h^{\frac{1}{2}} - u_h^0}{\frac{\Delta t}{2}} v_h + a_h^0(u_h^0, v_h) + \gamma_s s_h^{0,\frac{1}{2}}(u_h^{\frac{1}{2}}, v_h) = \int_{\Omega_h^0} f_h^0 v_h, \quad \forall v_h \in V_h^0 \tag{4.16}$$

γ_s is a stabilization parameter. In the second substep is an implicit Euler time step. We extend $\Omega_h^{\frac{1}{2}}$ to its $\frac{\delta_h}{2}$-neighborhood $\mathcal{O}^{\frac{1}{2}}_{\frac{\delta_h}{2},\mathcal{T}}$. This discrete extension can be realized by a stablization term $s^{\frac{1}{2},\frac{1}{2}}(u_h^1, v_h)$ and $s^{\frac{1}{2},\frac{1}{2}}(u_h^1, v_h)$ guarantees that u_h^1 is well defined in $\{\Omega_h^{\frac{1}{2}}, \Omega_h^1\}$.

$$\int_{\Omega_h^1} \frac{u_h^1 - u^{\frac{1}{2}}}{\frac{\Delta t}{2}} v_h + a_h^1(u_h^1, v_h) + \gamma_s s_h^{\frac{1}{2},\frac{1}{2}}(u_h^1, v_h) = \int_{\Omega_h^1} f_h^1 v_h, \quad \forall v_h \in V_h^1, \tag{4.17}$$

Hence, by the analogy the full discretization of first characterization of trapezoidal rule reads as: Suppose $u_h^0 \in V_h^0$, find $u_h^n \in V_h^n$, for n=1, 2, ..., N there holds

$$\int_{\Omega_h^{n-1}} \frac{u_h^{n'} - u_h^{n-1}}{\frac{\Delta t}{2}} v_h + a_h^{n-1}(u_h^{n-1}, v_h) + \gamma_s s_h^{n-1,\frac{1}{2}}(u_h^{n'}, v_h) = \int_{\Omega_h^{n-1}} f_h^{n-1} v_h, \ \forall v_h \in V_h^{n-1}, \tag{4.18a}$$

$$\int_{\Omega_h^n} \frac{u_h^n - u^{n'}}{\frac{\Delta t}{2}} v_h + a_h^n(u_h^n, v_h) + \gamma_s s_h^{n',\frac{1}{2}}(u_h^n, v_h) = \int_{\Omega_h^n} f_h^n v_h, \ \forall v_h \in V_h^n. \tag{4.18b}$$

where $n' := \frac{2n-1}{2}$ and $a_h^{n-1}(\cdot,\cdot)$ and $a_h^n(\cdot,\cdot)$ are discrete bilinear forms defined as (4.9).

4.4.2 Full discretization of second characterization

The second characterization of trapezodial rule is a trapezodial rule based on an averaged Euler steps. The numerical method is based on (3.37). When n=1, for the explicit Euler time stepping, we extend Ω_h^0 to its δ_h-neighborhood $\mathcal{O}^0_{\delta_h,\mathcal{T}}$, i.e. we extend Ω_h^0 by thinkness δ_h so that Ω_h^0, Ω_h^1 and Ω_h^2 are subsets of $\mathcal{O}^0_{\delta_h,\mathcal{T}}$. This discrete extendsion can be realized by a stabilization term $s_h^{0,2}(u_h^{1,e}, v_h)$ and $s_h^{0,2}(u_h^{1,e}, v_h)$ guarantees that $u_h^{1,e}$ is well defined on $\{\Omega_h^0, \Omega_h^1, \Omega_h^2\}$. Then suppose $u_h^0 \in V_h^0$ we have

$$\int_{\Omega_h^0} \frac{u_h^{1,e} - u_h^0}{\Delta t} v_h + a_h^0(u_h^0, v_h) + \gamma_s s_h^{0,2}(u_h^{1,e}, v_h) = \int_{\Omega_h^0} f_h^0 v_h, \ \forall v_h \in V_h^0. \tag{4.19}$$

For the implicit Euler time stepping, we extend Ω_h^1 to $\mathcal{O}^1_{\delta_h,\mathcal{T}}$ by thinkness δ_h s.t. Ω_h^1 and Ω^2 are subsets of $\mathcal{O}^1_{\delta_h,\mathcal{T}}$, this discrete extension can be realized by a stabilization term $s_h^{1,1}(u_h^{1,i}, v_h)$ and $s_h^{1,1}(u_h^{1,i}, v_h)$ guarantees that $u_h^{1,i}$ is well defined on $\{\Omega_h^1, \Omega_h^2\}$. Then suppose $u_h^0 \in V_h^0$ we have

$$\int_{\Omega_h^1} \frac{u_h^{1,i} - u_h^0}{\Delta t} v_h + a_h^1(u_h^{1,i}, v_h) + \gamma_s s_h^{1,1}(u_h^{1,i}, v_h) = \int_{\Omega_h^1} f_h^1 v_h, \ \forall v_h \in V_h^1. \tag{4.20}$$

Therefore, we obtain u_h^1 by a computation of average of (4.19) and (4.20)

$$u_h^1 = \frac{u_h^{1,e} + u_h^{1,i}}{2}, \quad \text{in} \quad \mathcal{O}^0_{2\delta_h,\mathcal{T}} \bigcap \mathcal{O}^1_{\delta_h,\mathcal{T}}. \tag{4.21}$$

Hence, by the analogy the full discretization for the second characterization of trapezodial rule reads as : Suppose $u_h^0 \in V_h^0$, find $u_h^n \in V_h^n$, for n=1, 2, ..., N there holds

$$\int_{\Omega_h^{n-1}} \frac{u_h^{n,e} - u_h^{n-1}}{\Delta t} v_h + a_h^{n-1}(u_h^{n-1}, v_h) + \gamma_s s_h^{n-1,2}(u_h^{n,e}, v_h) = \int_{\Omega_h^{n-1}} f_h^{n-1} v_h, \ \forall v_h \in V_h^{n-1}, \tag{4.22a}$$

$$\int_{\Omega_h^n} \frac{u_h^{n,i} - u_h^{n-1}}{\Delta t} v_h + a_h^n(u_h^{n,i}, v_h) + \gamma_s s_h^{n,1}(u_h^{n,i}, v_h) = \int_{\Omega_h^1} f_h^n v_h, \ \forall v_h \in V_h^n, \tag{4.22b}$$

where $a_h^{n-1}(\cdot,\cdot)$ and $a_h^n(\cdot,\cdot)$ are disrete bilinear forms for diffusion and convection parts defined as (4.9),

$$u_h^n = \frac{u_h^{n,e} + u_h^{n,i}}{2}, \quad \text{in} \quad \mathcal{O}_{2\delta_h,\mathcal{T}}^{n-2} \bigcap \mathcal{O}_{\delta_h,\mathcal{T}}^{n-1}. \tag{4.23}$$

4.4.3 Full discretization of third characterization

Third characterization is a naive extension version of (3.25). The numerical method is based on (3.39). The full discretiztion for the third characterization of trapezodial rule is reads as: Suppose $u_h^0 \in V_h^0$, find $u_h^n \in V_h^n$, for n=1, 2, ..., N there holds

$$\int_{\Omega_h^n} \frac{u_h^n - u_h^{n-1}}{\frac{\Delta t}{2}} v_h + a_h^n(u_h^n, v_h) + a_h^{n-1}(u_h^{n-1}, v_h) + \gamma_s s_h^n(u_h^n, v_h) = \int_{\Omega_h^n} f_h^n v_h + f_h^{n-1} v_h, \ \forall v_h \in V_h^n, \tag{4.24}$$

where $a_h^n(\cdot,\cdot)$ and $a_h^{n-1}(\cdot,\cdot)$ are the disrete bilinear forms for diffusion and convection parts defined as (4.9).

4.5 Full discretezation of midpoint rule

As we have discussed in chapter 3, midpoint rule is also based on subsequent Euler half steps. The numerical method is based on (3.45). When $n = 1$, suppose $u_h^0 \in V_h^0$, applying (3.8) we have

$$\int_{\Omega_h^{\frac{1}{2}}} \frac{u_h^{\frac{1}{2}} - u_h^0}{\frac{\Delta t}{2}} v_h + a_h^{\frac{1}{2}}(u_h^{\frac{1}{2}}, v_h) + \gamma_s s_h^{\frac{1}{2}}(u_h^{\frac{1}{2}}, v_h) = \int_{\Omega_h^{\frac{1}{2}}} f_h^{\frac{1}{2}} v_h, \ \forall v_h \in V_h^{\frac{1}{2}}. \tag{4.25}$$

There is no extension within this time step, the stabilization term $s_h^{\frac{1}{2}}(u_h^{\frac{1}{2}}, v_h)$ just guarantees the stability. Next, we extend $\Omega_h^{\frac{1}{2}}$ to its $\frac{\delta_h}{2}$-neighborhood $\mathcal{O}_{\delta_h,\mathcal{T}}^{\frac{1}{2}}$ by a thinkness of $\frac{\delta_h}{2}$. This discrete extension can be realized by a stabilization term $s_h^{\frac{1}{2},1}(u_h^1, v_h)$ s.t. $\Omega_h^{\frac{3}{2}} \subset \mathcal{O}_{\delta_h,\mathcal{T}}^{\frac{1}{2}}$, then applying (3.22) we have

$$\int_{\Omega_h^{\frac{1}{2}}} \frac{u_h^1 - u_h^{\frac{1}{2}}}{\frac{\Delta t}{2}} + a_h^{\frac{1}{2}}(u_h^{\frac{1}{2}}, v_h) + \gamma_s s_h^{\frac{1}{2},1}(u_h^1, v_h) = \int_{\Omega_h^{\frac{1}{2}}} f_h^{\frac{1}{2}} v_h, \ \forall v_h \in V_h^{\frac{1}{2}}. \tag{4.26}$$

The stabilization term $s_h^{\frac{1}{2},1}(u_h^1, v_h)$ guarantees that u_h^1 is well defined in $\Omega_h^{\frac{3}{2}}$.

Combine the (4.25) and (4.26) we obatin

$$\int_{\Omega_h^{\frac{1}{2}}} \frac{u_h^1 - u_h^0}{\Delta t} + a_h^{\frac{1}{2}}(u_h^{\frac{1}{2}}, v_h) + \frac{1}{2}(\gamma_s s_h^{\frac{1}{2}}(u_h^{\frac{1}{2}}, v_h) + \gamma_s s_h^{\frac{1}{2},1}(u_h^1, v_h)) = \int_{\Omega_h^{\frac{1}{2}}} f_h^{\frac{1}{2}} v_h, \; \forall v_h \in V_h^{\frac{1}{2}}. \tag{4.27}$$

Hence, by the analogy the full discretization for midpoint rule reads as :

$$\int_{\Omega_h^{n'}} \frac{u_h^n - u_h^{n-1}}{\Delta t} v_h + a_h^{n'}(u_h^{n'}, v_h) + \frac{1}{2}(\gamma_s s_h^{n'}(u_h^{n'}, v_h) + \gamma_s s_h^{n',1}(u_h^n, v_h)) = \int_{\Omega_h^{n'}} f_h^{n'} v_h, \quad \forall v_h \in V_h^{n'}, \tag{4.28}$$

where $n' := \frac{2n-1}{2}$ and $a_h^{n'}(\cdot,\cdot)$ is the discrete bilinear form for diffusion and convection parts defined as (4.9). The numerical solution $u_h^n \in \mathcal{O}_{2\delta_h,\mathcal{T}}^{n-1} \bigcap \mathcal{O}_{\delta_h,\mathcal{T}}^{\frac{2n-1}{2}}$. We note that the formulation (4.28) is not directly implementable. The numerical implementation of (4.28) can be via two half steps.

Chapter 5

Stability analysis of the fully discrete methods

In the analysis we make use of time-dependent domains stemming from extensions. To this end,we define a strip in each time step which is *sharp* and can include full and cut elements:

$$
\begin{aligned}
S^{\pm}_{l\cdot\delta_h}(\Omega^n_h) &:= \{\mathbf{x} \in \tilde{\Omega} : \text{dist}(\mathbf{x}, \Gamma^n_h) \le l \cdot \delta_h\} \\
S^{+}_{l\cdot\delta_h}(\Omega^n_h) &:= \{\mathbf{x} \in \tilde{\Omega}\backslash\Omega^n_h : \text{dist}(\mathbf{x}, \Gamma^n_h) \le l \cdot \delta_h\}.
\end{aligned}
\tag{5.1}
$$

We notice that $S^{\pm}_{l\cdot\delta_h}$ includes points from the interior of Ω^n_h as well as points from the outside Ω^n_h. All elements with parts in $S^{\pm}_{l\cdot\delta_h}$ are collected in $\mathcal{T}^n_{S^{\pm}}$. In the analysis we ask for δ large enough so that

$$
\begin{aligned}
\mathcal{O}^{n,l}_{\delta_h,\mathcal{T}} &\subset \mathcal{O}_{l\cdot\delta}(\Omega^n), \\
\Omega^n_h &\subset \mathcal{O}_{l\cdot\delta}(\Omega_t), \quad t \in I_n.
\end{aligned}
\tag{5.2}
$$

where we recall the definition of $\mathcal{O}^{n,l}_{\delta_h,\mathcal{T}}$ and $\mathcal{O}_{l\cdot\delta}(\Omega^n)$ from chapter 2.

We specify

$$\delta_h = c_{\delta_h}\mathbf{w}_\infty\Delta t, \quad \text{with} \quad 1 < c_{\delta_h} < c_\delta. \tag{5.3}$$

where $\mathbf{w}_\infty$ is the (global in time) maximum absolute value of the velocity. With this condition there holds

$$c_{\delta_h} \text{large enough} \Rightarrow \Omega^n_h \subset \mathcal{O}^{n-l,l}_{\delta_h,\mathcal{T}}, \quad l \in \{1,2,...,n\},\ n \in \{1,2,...,N\}. \tag{5.4}$$

5.1 Stability of discrete extension

In ([1], section 5.3) C.Lehrenfeld and M.A.Olshanskii have proven the numerical stability of discrete extensions through a stabilization term $s^n_h(\cdot,\cdot)$ by the *Lemma* as follows:

Lemma 2. *Let $T_1 \in \mathcal{T}^n_{S\pm}$ and $T^2 \in \mathcal{T}^n_\delta$, $T_1 \neq T_2$ so that for $F = \overline{T_1} \bigcap \overline{T_2}$ there holds* $\text{meas}_{d-1}(F) > 0$. *Then we have for $u|_{T_i} \in \mathbb{P}_k(T_i),\ i{=}1,2$ that there holds*

$$\begin{aligned} \|u\|^2_{T_1} &\leq \|u\|^2_{T_2} + s_{h,F}(u,u), \\ \|\nabla u\|^2_{T_1} &\leq \|\nabla u\|^2_{T_2} + \frac{1}{h^2} s_{h,F}(u,u). \end{aligned} \tag{5.5}$$

In order to apply this stabilizing mechanism globally, they make a significant assumption on the meshes $\mathcal{T}^n_{S+}$ and $\mathcal{T}^n_\delta$.
Assumption 1. *To every element in $\mathcal{T}^n_{S+}$ we require an element in $\mathcal{T}^n_\delta \backslash \mathcal{T}^n_{S+}$ that can be reached by repeatedly passing through facets in F^n_h. We assume that there is a mapping that maps every element $T \in \mathcal{T}^n_{S+}$ to such a path with the following properties. The number of facets passed through during this path is bounded by $K \leq (1 + \frac{\delta_h}{h})$. Further, every uncut element $T \in \mathcal{T}^n_\delta \backslash \mathcal{T}^n_{S+}$ is the final element of such a path in at most M of these paths where M is a number that is bounded independent of h and Δt.*

This is a reasonable assumption if the boundary Γ^n is sufficiently well-resolved by the mesh, the comment on this assumption is in Remark 5.2 in [1]. This assumption provides some important results (*Lemma 5.2*, *Lemma 5.3*, *Lemma 5.4 and Lemma 5.5* in [1]) which are useful mathmatical tools to prove the numerical stability of fully disrete methods.

5.2 Stability analysis

5.2.1 Stability analysis of full discretization of BDF2 method in time

For the simplicity, in the following analysis f is set to be 0. *Theorem 5.1* in [1] proves a stability of a full discretization based on the stabilized implicit Euler time stepping method. Using the result of *Theorem 5.1* we give a stability analysis on a full discretization based on the stabilized BDF2 time stepping method. First, we introduce two important results from [1]
Lemma 3. *Using Assumption 1, for $u \in V^n_h$ and $\forall \varepsilon > 0$ there holds*

$$\|u\|^2_{\mathcal{S}^+_{\delta_h}(\Omega^n_h)} \leq \delta_h(1+\varepsilon^{-1})\|u\|^2_{\Omega^n_h} + \delta_h \varepsilon \|\nabla u\|^2_{\Omega^n_h} + \delta_h K((1+\varepsilon^{-1})h^2 + \varepsilon) s^n_h(u,u). \tag{5.6}$$

As a direct consequence we have for a constant c_{L_2} independent of h and Δt

$$\|u\|^2_{\mathcal{O}_{\delta_h}(\Omega^n_h)} \leq (1 + c_{L_{2a}}(\varepsilon)\Delta t)\|u\|^2_{\Omega^n_h} + c_{L_{2b}}\nu(\varepsilon)\|\nabla u\|^2_{\Omega^n_h} + c_{L_{2c}}(\varepsilon,h)\Delta t K s^n_h(u,u). \tag{5.7}$$

where

$$
\begin{aligned}
c_{L_{2a}}(\varepsilon) &= c_{L_2} c_{\delta_h} \mathbf{w}^{\mathbf{n}}_{\infty}(1+\varepsilon^{-1}),\\
c_{L_{2b}}(\varepsilon) &= c_{L_2} c_{\delta_h} \mathbf{w}^{\mathbf{n}}_{\infty}\frac{\varepsilon}{\nu},\\
c_{L_{2c}}(\varepsilon,h) &= c_{L_2} c_{\delta_h} \mathbf{w}^{\mathbf{n}}_{\infty}(\varepsilon + h^2 + h^2\varepsilon^{-1}).
\end{aligned} \tag{5.8}
$$

Remark 18. *Unique solvability Similarity to Lemma 3.1 in [1] we can easily check that*

$$
a_h^n(u_h, u_h) \geq \frac{\alpha}{2}\|\nabla u_h\|^2_{\Omega_h^n} - \xi_h \|u_h\|^2_{\Omega_h^n}, \tag{5.9}
$$

if

$$
\Delta t < \xi_h^{-1} := 2(\left\|\mathrm{div}(\mathbf{w}^e)\right\|_{L^\infty(\Omega_h^n)} + \alpha + c^2_{\Omega_h^n}\|\mathbf{w}^e\cdot\mathbf{n}\|_{L^\infty(\Omega_h^n)}/4\alpha)^{-1}. \tag{5.10}
$$

Now, we give a stability analysis on the full discretization based on the stabilized BDF2 time stepping method.

Theorem 1. *Under Assumption 1, we assume that u_h^0 and u_h^{-1} which is given in advanced are given as functions on Ω_h^1, c_γ large enough and Δt sufficiently small, the numerical solution of (4.12) satisfies the following estimate*

$$
\begin{aligned}
\left\|u_h^k\right\|^2_{\Omega_h^k} + \Delta t \sum_{n=2}^{k}(\nu\left\|\nabla u_h^{n-1}\right\|^2_{\Omega^{n-1}} + 2\gamma_s \Delta t s_h^n(u_h^n, u_h^n)) \leq \exp(c_{T_1} t_k)(\left\|u_h^0\right\|^2_{\Omega_h^1} + \left\|2u_h^0 - u_h^{-1}\right\|^2_{\Omega_h^1})\\
+ \Delta t|||u_h^0|||_0,
\end{aligned} \tag{5.11}
$$

with c_{T_1} independent of h, Δt. The norm $|||\cdot|||_0$ is defined as

$$
|||v|||_0 := (\nu\|\nabla v\|^2_{\Omega_h^0} + c\gamma_s s_h^{0,0}(v,v))^{\frac{1}{2}}. \tag{5.12}
$$

Proof. With the relationship

$$
2a(3a - 4b + c) = |a|^2 + |2a-b|^2 + |a - 2b + c|^2 - |b|^2 - |2b - c|^2 \tag{5.13}
$$

we test (4.12) with $2u_h^n$ and multiply by $2\Delta t$ which yields

$$\begin{aligned}&\|u_h^n\|^2_{\Omega_h^n} + \left\|2u_h^n - u_h^{n-1}\right\|^2_{\Omega_h^n} + \left\|u_h^n - 2u_h^{n-1} + u_h^{n-2}\right\|^2_{\Omega_h^n} + 4\Delta t a_h^n(u_h, u_h) + 4\Delta t \gamma_s s_h^{n,2}(u_h, u_h)\\ &= \left\|u_h^{n-1}\right\|^2_{\Omega_h^n} + \left\|2u_h^{n-1} - u_h^{n-2}\right\|^2_{\Omega_h^n}.\end{aligned} \tag{5.14}$$

Since $\left\|u_h^n - 2u_h^{n-1} + u_h^{n-2}\right\|^2_{\Omega_h^n} > 0$, we have

$$\begin{aligned}&\|u_h^n\|^2_{\Omega_h^n} + \left\|2u_h^n - u_h^{n-1}\right\|^2_{\Omega_h^n} + 4\Delta t a_h^n(u_h, u_h) + 4\Delta t \gamma_s s_h^{n,2}(u_h, u_h)\\ &\leq \left\|u_h^{n-1}\right\|^2_{\Omega_h^n} + \left\|2u_h^{n-1} - u_h^{n-2}\right\|^2_{\Omega_h^n}.\end{aligned} \tag{5.15}$$

Using the lower bound on $a_h^n(\cdot,\cdot)$ which is given in Remark 18 we have

$$\begin{aligned}&(1 - 4\Delta t \xi_h)\|u_h^n\|^2_{\Omega_h^n} + \left\|2u_h^n - u_h^{n-1}\right\|^2_{\Omega_h^n} + 2\nu\Delta t\|\nabla u_h^n\|^2_{\Omega_h^n} + 4\Delta t \gamma_s s_h^{n,2}(u_h, u_h)\\ &\leq \left\|u_h^{n-1}\right\|^2_{\Omega_h^n} + \left\|2u_h^{n-1} - u_h^{n-2}\right\|^2_{\Omega_h^n}.\end{aligned} \tag{5.16}$$

To go from $\left\|u_h^{n-1}\right\|_{\Omega_h^n}$ to $\left\|u_h^{n-1}\right\|_{\Omega_h^{n-1}}$ (and similarity for $\left\|2u_h^{n-1} - u_h^{n-2}\right\|$) use Lemma 2 we obtain

$$\begin{aligned}&(1 - 4\Delta t \xi_h)\|u_h^n\|^2_{\Omega_h^n} + \left\|2u_h^n - u_h^{n-1}\right\|^2_{\Omega_h^n} + 2\nu\Delta t\|\nabla u_h^n\|^2_{\Omega_h^n} + 4\Delta t \gamma_s s_h^{n,2}(u_h, u_h)\\ &\leq \left\|u_h^{n-1}\right\|^2_{\Omega_h^n} + \left\|2u_h^{n-1} - u_h^{n-2}\right\|^2_{\Omega_h^n} \leq \left\|u_h^{n-1}\right\|^2_{\mathcal{O}_\delta(\Omega_h^{n-1})} + \left\|2u_h^{n-1} - u_h^{n-2}\right\|^2_{\mathcal{O}_\delta(\Omega_h^{n-1})}\\ &\leq (1 + c_{L_{2a}}(\varepsilon)\Delta t)\left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} + c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} + c_{L_{2c}}(\varepsilon, h)\Delta t K s_h^{n-1,1}(u_h^{n-1}, u_h^{n-1})\\ &\quad + (1 + c_{L_{2a}}(\varepsilon)\Delta t)\left\|2u_h^{n-1} - u_h^{n-2}\right\|^2_{\Omega_h^{n-1}} + c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla(2u_h^{n-1} - u^{n-2})\right\|^2_{\Omega_h^{n-1}}\\ &\quad + c_{L_{2c}}(\varepsilon, h)\Delta t K s_h^{n-1,1}(2u_h^{n-1} - u_h^{n-2}, 2u_h^{n-1} - u_h^{n-2}).\end{aligned} \tag{5.17}$$

Since

$$\begin{aligned}&s_h^{n-1,1}(2u_h^{n-1} - u_h^{n-2}, 2u_h^{n-1} - u_h^{n-2})\\ &= 4s_h^{n-1,1}(u_h^{n-1}, u_h^{n-1}) - 4s_h^{n-1,1}(u_h^{n-1}, u_h^{n-2}) + s_h^{n-1,1}(u_h^{n-2}, u_h^{n-2})\\ &\leq 4s_h^{n-1,1}(u_h^{n-1}, u_h^{n-1}) + 2\gamma s_h^{n-1}(u_h^{n-1}, u_h^{n-1}) + \frac{2}{\gamma}s_h^{n-1,1}(u_h^{n-2}, u_h^{n-2}) + s_h^{n-1,1}(u_h^{n-2}, u_h^{n-2}),\end{aligned} \tag{5.18}$$

with Cauchy-Schwarz inequality.

Let $\gamma = \frac{1}{2}$ which yields

$$s_h^{n-1,1}(2u_h^{n-1} - u_h^{n-2}, 2u_h^{n-1} - u_h^{n-2}) \leq 5s_h^{n-1,1}(u_h^{n-1}, u_h^{n-1}) + 5s_h^{n-1,1}(u_h^{n-2}, u_h^{n-2}). \tag{5.19}$$

Similarly,

$$\left\|\nabla(2u_h^{n-1} - u_h^{n-2})\right\|_{\Omega_h^{n-1}}^2 \leq 5\left\|\nabla u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 + 5\left\|\nabla u_h^{n-2}\right\|_{\Omega_h^{n-1}}^2 \tag{5.20}$$

Use the Lemma 5.2 in [1]

$$\left\|\nabla u_h^{n-2}\right\|_{\Omega_h^{n-1}}^2 \leq \left\|\nabla u_h^{n-2}\right\|_{\Omega_h^{n-2}} + Ks_h^{n-1,1}(u_h^{n-2}, u_h^{n-2}) \tag{5.21}$$

Hence

$$\begin{aligned}\left\|\nabla(2u_h^{n-1} - u_h^{n-2})\right\|_{\Omega_h^{n-1}}^2 &\leq 5\left\|\nabla u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 + 5\left\|\nabla u_h^{n-2}\right\|_{\Omega_h^{n-1}}^2 \\ &\leq 5\left\|\nabla u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 + 5\left\|\nabla u_h^{n-2}\right\|_{\Omega_h^{n-2}} + 5Ks_h^{n-1,1}(u_h^{n-2}, u_h^{n-2})\end{aligned} \tag{5.22}$$

Substitute (5.19) and (5.22) into (5.17) we obtain

$$\begin{aligned}&(1 - 4\Delta t\xi_h)\|u_h^n\|_{\Omega_h^n}^2 + \left\|2u_h^n - u_h^{n-1}\right\|_{\Omega_h^n}^2 + 2\nu\Delta t\|\nabla u_h^n\|_{\Omega_h^n}^2 + 4\Delta t\gamma_s s_h^{n,2}(u_h, u_h) \\ \leq &\left\|u_h^{n-1}\right\|_{\Omega_h^n}^2 + \left\|2u_h^{n-1} - u_h^{n-2}\right\|_{\Omega_h^n}^2 \leq \left\|u_h^{n-1}\right\|_{\mathcal{O}_\delta(\Omega_h^{n-1})}^2 + \left\|2u_h^{n-1} - u_h^{n-2}\right\|_{\mathcal{O}_\delta(\Omega_h^{n-1})}^2 \\ \leq &(1 + c_{L_{2a}}(\varepsilon)\Delta t)\left\|u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 + 6c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 + 6c_{L_{2c}}(\varepsilon, h)\Delta tKs_h^{n-1,1}(u_h^{n-1}, u_h^{n-1}) \\ &+ (1 + c_{L_{2a}}(\varepsilon)\Delta t)\left\|2u_h^{n-1} - u_h^{n-2}\right\|_{\Omega_h^{n-1}}^2 + 5c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n-2}\right\|_{\Omega_h^{n-2}}^2 \\ &+ 5(c_{L_{2c}}(\varepsilon, h) + c_{L_{2b}}(\varepsilon))\Delta tKs_h^{n-1,1}(u_h^{n-2}, u_h^{n-2}).\end{aligned} \tag{5.23}$$

Set $c_{L_{2c}}(\varepsilon, h) + c_{L_{2b}}(\varepsilon) \leq 2c_{L_{2c}}$ yields the last term of (5.23)

$$5(c_{L_{2c}}(\varepsilon, h) + c_{L_{2b}}(\varepsilon))\Delta tKs_h^{n-1,1}(u_h^{n-2}, u_h^{n-2}) \leq 10c_{L_{2c}}\Delta tKs_h^{n-1,1}(u_h^{n-2}, u_h^{n-2}) \tag{5.24}$$

We choose $\varepsilon \leq \frac{\nu}{12c_{L_2}c_{\delta_h}\mathbf{w}_\infty^\mathbf{n}}$ so that $c_{L_{2b}}(\varepsilon) \leq \frac{1}{12}$, $c_{L_{2a}}(\varepsilon)$ and $c_{L_{2c}}(\varepsilon, h)$ are bounded independent of h and Δt. We define

$$(1 - 4\Delta t\xi_h)q^n := (1 - 4\Delta t\xi_h)\|u_h^n\|_{\Omega_h^n}^2 + \left\|2u_h^n - u_h^{n-1}\right\|_{\Omega_h^n}^2. \tag{5.25}$$

Then (5.23) can be written as

$$\begin{aligned}
&(1-4\Delta t\xi_h)q^n + 2\nu\Delta t\|\nabla u_h^n\|^2_{\Omega_h^n} + 4\Delta t\gamma_s s_h^{n,2}(u_h,u_h)\\
\le\ &(1+c_{L_{2a}}(\varepsilon)\Delta t)q^{n-1} + \frac{\Delta t}{2}\nu\left\|\nabla u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} + \frac{\Delta t}{2}\nu\left\|\nabla u_h^{n-2}\right\|^2_{\Omega_h^{n-2}}\\
&+ 6c_{L_{2c}}(\varepsilon,h)\Delta tKs_h^{n-1,1}(u_h^{n-1},u_h^{n-1}) + 10c_{L_{2c}}\Delta tKs_h^{n-2,2}(u_h^{n-2},u_h^{n-2}).
\end{aligned} \tag{5.26}$$

We assume $\gamma_s \ge 8c_{L_{2c}}K$ and summe up over $n=1,2,...,k,\ k\le N$ yields

$$\begin{aligned}
&(1-4\Delta t\xi_h)q^k + \nu\Delta t\sum_{n=1}^{k}\|\nabla u_h^n\|^2_{\Omega_h^n} + 2\Delta t\gamma_s\sum_{n=1}^{k}s_h^{n,2}(u_h,u_h)\\
\le&\left\|u_h^0\right\|^2_{\Omega_h^1} + \left\|2u^0-u^{-1}\right\|^2_{\Omega_h^1} + +\Delta t|||u_h^0|||_0 + c_{T_1}\Delta t\sum_{n=1}^{k-1}q^n
\end{aligned} \tag{5.27}$$

Applying Gronwall's Lemma with $\xi_h\Delta t \le \frac{1}{8}$ we obtain the result with $c_{T_1} = c_{L_{2a}} + 2\xi_h$. □

5.2.2 Stability analysis of full discretization of the explicit Euler method in time

Theorem 2. *Under Assumption 1 we assume u_h^0 well-definded in Ω_h^0 and $\Delta t \le C_\nu h^2$ for a C_ν only depending on ν, the numerical solution of (4.15) satisfies the following estimate*

$$\left\|u_h^k\right\|^2_{\Omega_h^{k-1}} + \nu\Delta t\sum_{n=1}^{k}\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} + 2\gamma_s\Delta t\sum_{n=1}^{k}s_h^{n-1}(u_h^n,u_h^n) \le \exp(c_{T_1}t_k)\left\|u_h^0\right\|^2_{\Omega_h^0} \tag{5.28}$$

for a c_{T_1} only depending on ξ_h.

Proof. Together with the relationship

$$2a(a-b) = |a|^2 + |a-b|^2 - |b|^2 \tag{5.29}$$

we test (4.15) with $2u^n$ and multiply by $2\Delta t$ which yields

$$\|u_h^n\|^2_{\Omega_h^{n-1}} + \left\|u_h^n - u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} + 2\Delta ta_h^{n-1}(u_h^{n-1},u_h^n) + 2\Delta t\gamma_s s_h^{n-1}(u_h^n,u_h^n) = \left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}}. \tag{5.30}$$

Use the lower bound on $a^{n-1}(\cdot,\cdot)$ and Jensen inequality we have

$$
\begin{aligned}
a_h^{n-1}(u_h^{n-1},u_h^n) &= a_h^{n-1}(u_h^n,u_h^n) + a_h^{n-1}(u_h^{n-1}-u_h^n,u_h^n) \\
&\geq \frac{\nu}{2}\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} - \xi_h\|u_h^n\|^2_{\Omega_h^{n-1}} - \left\|a_h^{n-1}\right\|\left\|u_h^{n-1}-u_h^n\right\|_{\Omega_h^{n-1}}\|\nabla u_h^n\|_{\Omega_h^{n-1}} \\
&\geq \frac{\nu}{2}\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} - \xi_h\|u_h^n\|^2_{\Omega_h^{n-1}} - c_h\left\|u_h^{n-1}-u_h^n\right\|_{\Omega_h^{n-1}}\|\nabla u_h^n\|_{\Omega_h^{n-1}} \\
&\geq \frac{\nu}{2}\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} - \xi_h\|u_h^n\|^2_{\Omega_h^{n-1}} - Ch^{-1}\left\|u_h^n-u_h^{n-1}\right\|_{\Omega_h^{n-1}}\|\nabla u_h^n\|_{\Omega_h^{n-1}} \\
&\geq \frac{\nu}{2}\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} - \xi_h\|u_h^n\|^2_{\Omega_h^{n-1}} - \frac{2C^2h^{-2}}{\nu}\left\|u_h^n-u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} - \frac{\nu}{4}\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} \\
&\geq \frac{\nu}{4}\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} - \xi_h\|u_h^n\|^2_{\Omega_h^{n-1}} - \frac{2C^2h^{-2}}{\nu}\left\|u_h^n-u_h^{n-1}\right\|^2_{\Omega_h^{n-1}}
\end{aligned}
\tag{5.31}
$$

with $c_h := \left\|a_h^{n-1}\right\| = \sup_{v_h,w_h\in V_h^{n-1}} \frac{a_h^{n-1}(v_h,\nabla w_h)}{\|v_h\|_{L^2}\|\nabla w_h\|_{L^2}} \leq Ch^{-1}$.

Substituting (5.31) into (5.30) we obtain

$$
\begin{aligned}
&(1-2\xi_h\Delta t)\|u_h^n\|^2_{\Omega_h^{n-1}} + (1-\frac{2C^2h^{-2}}{\nu}\Delta t)\left\|u_h^n-u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} + \frac{\nu}{2}\Delta t\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} \\
&+ 2\Delta t\gamma_s s_h^{n-1}(u_h^n,u_h^n) \leq \left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}}.
\end{aligned}
\tag{5.32}
$$

We set $1-\frac{2C^2h^{-2}}{\nu}\Delta t \geq 0$ i.e. $\Delta t \leq \frac{\nu}{2C^2h^{-2}} = C_\nu h^2$ s.t.

$$
(1-\frac{2C^2h^{-2}}{\nu}\Delta t)\left\|u_h^n-u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} \geq 0 \tag{5.33}
$$

and together we have

$$
(1-2\xi_h\Delta t)\|u_h^n\|^2_{\Omega_h^{n-1}} + \frac{\nu}{2}\Delta t\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} + 2\gamma_s\Delta t s_h^{n-1}(u_h^n,u_h^n) \leq \left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}}, \tag{5.34}
$$

Summing up over $n=1,2,...,k,\ k\leq N$ we obtain

$$
\begin{aligned}
&\left\|u_h^k\right\|^2_{\Omega_h^{k-1}} + \frac{\nu}{2}\Delta t\sum_{n=1}^{k}\|\nabla u_h^n\|^2_{\Omega_h^{n-1}} + 2\gamma_s\Delta t\sum_{n=1}^{k}s_h^{n-1}(u_h^n,u_h^n) \\
&\leq \left\|u_h^0\right\|^2_{\Omega_h^0} + 2\xi_h\Delta t\sum_{n=1}^{k}\|u_h^n\|^2_{\Omega_h^{n-1}}
\end{aligned}
\tag{5.35}
$$

Applying Gronwall's Lemma with $\xi_h\Delta t \leq \frac{1}{8}$ we obtain the result with $c_{T_1} := 2\xi_h$. □

5.2.3 Stability analysis of full discretization of trapezoidal rules

Stability analysis of first characterization

Theorem 3. *Under Assumption 1 we assume u_h^0 well-defended in Ω_h^0 and $\Delta t \le \min\{\frac{1}{c_{L_{2a}}(\varepsilon)}, C_\nu h^2\}$, the numerical solution of (4.18) satisfies the following estimate*

$$\left\|u_h^k\right\|_{\Omega_h^k}^2 + \frac{\nu}{4}\Delta t \sum_{n=1}^{k} \|\nabla u_h^n\|_{\Omega_h^n}^2 + \gamma_s \Delta t \sum_{n=1}^{k} (s_h^{n',\frac{1}{2}}(u_h^n, u_h^n) + k_1 s_h^{n-1,\frac{1}{2}}(u_h^{n'}, u_h^{n'})) \le \exp(c_{T_1} t_k)\left\|u_h^0\right\|_{\Omega_h^0}^2 \tag{5.36}$$

for C_ν only depending on ν, c_{T_1} depending on ξ_h, $k' := \frac{2k-1}{2}$ and $n' := \frac{2n-1}{2}$.

Proof. We test (4.18a) with $2u_h^{n'}$ and multiply with $\frac{\Delta t}{2}$ we obtain

$$\left\|u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 + \left\|u_h^{n'} - u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 + \Delta t a_h^{n-1}(u_h^{n-1}, u_h^{n'}) + 2\Delta t \gamma_s s_h^{n-1,\frac{1}{2}}(u_h^{n'}, u_h^{n'}) = \left\|u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 \tag{5.37}$$

Then we test (4.18b) with $2u_h^n$ and multiply with $\frac{\Delta t}{2}$ we obtain

$$\|u_h^n\|_{\Omega_h^n}^2 + \left\|u_h^n - u_h^{n'}\right\|_{\Omega_h^n}^2 + \Delta t a_h^n(u_h^n, u_h^n) + 2\Delta t \gamma_s s_h^{n',\frac{1}{2}}(u_h^n, u_h^n) = \left\|u_h^{n'}\right\|_{\Omega_h^n}^2 \tag{5.38}$$

Combine (5.37) and (5.38) we have

$$\begin{aligned} &\|u_h^n\|_{\Omega_h^n}^2 + \left\|u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 + \left\|u_h^n - u_h^{n'}\right\|_{\Omega_h^n}^2 + \left\|u_h^{n'} - u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 + \Delta t(a_h^n(u_h^n, u_h^n) + a_h^{n-1}(u_h^{n-1}, u_h^{n'})) \\ &+ 2\Delta t \gamma_s (s_h^{n',\frac{1}{2}}(u_h^n, u_h^n) + s_h^{n-1,\frac{1}{2}}(u_h^{n'}, u_h^{n'})) = \left\|u_h^{n'}\right\|_{\Omega_h^n}^2 + \left\|u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 \end{aligned} \tag{5.39}$$

Using the lower bound on $a^n(\cdot,\cdot)$ we have

$$a_h^n(u_h^n, u_h^n) \ge \frac{\nu}{2}\|\nabla u_h^n\|_{\Omega_h^n}^2 - \xi_h \|u_h^n\|_{\Omega_h^n}^2 . \tag{5.40}$$

Using the lower bound on $a^{n-1}(\cdot,\cdot)$ we have

$$\begin{aligned} &a_h^{n-1}(u_h^{n-1}, u_h^{n'}) = a_h^{n-1}(u_h^{n'}, u_h^{n'}) + a_h^{n-1}(u_h^{n-1} - u_h^{n'}, u_h^{n'}) \\ &\ge \frac{\nu}{2}\left\|\nabla u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 - \xi_h \left\|u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 - \left\|a_h^{n-1}\right\|\left\|u_h^{n-1} - u_h^{n'}\right\|_{\Omega_h^{n-1}}\left\|\nabla u_h^{n'}\right\|_{\Omega_h^{n-1}} \\ &\ge \frac{\nu}{2}\left\|\nabla u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 - \xi_h \left\|u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 - c_h \left\|u_h^{n'} - u_h^{n-1}\right\|_{\Omega_h^{n-1}}\left\|\nabla u_h^{n'}\right\|_{\Omega_h^{n-1}} \\ &\ge \frac{\nu}{2}\left\|\nabla u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 - \xi_h \left\|u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 - \frac{2C^2h^{-2}}{\nu}\left\|u_h^{n'} - u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 - \frac{\nu}{4}\left\|\nabla u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 \\ &\ge \frac{\nu}{4}\left\|\nabla u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 - \xi_h \left\|u_h^{n'}\right\|_{\Omega_h^{n-1}}^2 - \frac{2C^2h^{-2}}{\nu}\left\|u_h^{n'} - u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2 \end{aligned} \tag{5.41}$$

with $c_h := \left\|a_h^{n-1}\right\| = \sup_{v_h, w_h \in V_h^{n-1}} \frac{a_h^{n-1}(v_h, \nabla w_h)}{\|v_h\|_{L^2}\|\nabla w_h\|_{L^2}} \leq Ch^{-1}$.

Substituting (5.40) and (5.41) into (5.39) we have

$$\begin{aligned}&(1-\Delta t\xi_h)(\|u_h^n\|^2_{\Omega_h^n} + \left\|u_h^{n'}\right\|^2_{\Omega_h^{n-1}}) + \frac{\nu}{4}\Delta t(\|\nabla u_h^n\|^2_{\Omega_h^n} + \left\|\nabla u_h^{n'}\right\|^2_{\Omega_h^{n-1}}) + \left\|u_h^n - u_h^{n'}\right\|^2_{\Omega_h^n}\\&+(1-\frac{2C^2h^{-2}}{\nu}\Delta t)\left\|u_h^{n'} - u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} + 2\Delta t\gamma_s(s_h^{n',\frac{1}{2}}(u_h^n, u_h^n) + s_h^{n-1,\frac{1}{2}}(u_h^{n'}, u_h^{n'}))\\&\leq \left\|u_h^{n'}\right\|^2_{\Omega_h^n} + \left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}}\end{aligned} \tag{5.42}$$

Let $1 - \frac{2C^2h^{-2}}{\nu}\Delta t \geq 0$ i.e. $\Delta t \leq \frac{\nu}{2C^2h^{-2}} = C_\nu h^2$ we have

$$(1-\frac{C^2h^{-2}}{2\nu}\Delta t)\left\|u_h^{n'} - u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} \geq 0. \tag{5.43}$$

Since $\left\|u_h^n - u_h^{n'}\right\|^2_{\Omega_h^n} \geq 0$, (5.42) can be written as

$$\begin{aligned}&(1-\Delta t\xi_h)(\|u_h^n\|^2_{\Omega_h^n} + \left\|u_h^{n'}\right\|^2_{\Omega_h^{n-1}}) + \frac{\nu}{4}\Delta t(\|\nabla u_h^n\|^2_{\Omega_h^n} + \left\|\nabla u_h^{n'}\right\|^2_{\Omega_h^{n-1}})\\&+2\Delta t\gamma_s(s_h^{n',\frac{1}{2}}(u_h^n, u_h^n) + s_h^{n-1,\frac{1}{2}}(u_h^{n'}, u_h^{n'})) \leq \left\|u_h^{n'}\right\|^2_{\Omega_h^n} + \left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}}\end{aligned} \tag{5.44}$$

From Lemma (2) we know that

$$\begin{aligned}\left\|u_h^{n'}\right\|^2_{\Omega_h^n} \leq \left\|u_h^{n'}\right\|^2_{\mathcal{O}_{\frac{\delta_h}{2}}(\Omega_h^{n'})} &\leq (1+c_{L_{2a}}(\varepsilon)\Delta t)\left\|u^{n'}\right\|^2_{\Omega_h^{n'}} + c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n'}\right\|^2_{\Omega_h^{n'}}\\&\quad + c_{L_{2c}}(\varepsilon,h)\Delta tKs_h^{n',\frac{1}{2}}(u_h^n, u_h^n)\\\left\|u_h^{n'}\right\|^2_{\Omega_h^{n'}} \leq \left\|u_h^{n'}\right\|^2_{\mathcal{O}_{\frac{\delta_h}{2}}(\Omega_h^{n-1})} &\leq (1+c_{L_{2a}}(\varepsilon)\Delta t)\left\|u_h^{n'}\right\|^2_{\Omega_h^{n-1}} + c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n'}\right\|^2_{\Omega_h^{n-1}}\\&\quad + c_{L_{2c}}(\varepsilon,h)\Delta tKs_h^{n-1,\frac{1}{2}}(u_h^{n'}, u_h^{n'})\end{aligned} \tag{5.45}$$

Rearranging (5.45) we obtain

$$\begin{aligned}\left\|u_h^{n'}\right\|^2_{\Omega_h^n} &\leq (1+c_{L_{2a}}(\varepsilon)\Delta t)\left\|u^{n'}\right\|^2_{\Omega_h^{n'}} + c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n'}\right\|^2_{\Omega_h^{n'}} + c_{L_{2c}}(\varepsilon,h)\Delta tKs_h^{n',\frac{1}{2}}(u_h^n, u_h^n)\\&\leq (1+c_{L_{2a}}(\varepsilon)\Delta t)^2\left\|u_h^{n'}\right\|^2_{\Omega_h^{n-1}} + (1+c_{L_{2a}}(\varepsilon)\Delta t)c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n'}\right\|^2_{\Omega_h^{n-1}}\\&\quad + (1+c_{L_{2a}}(\varepsilon)\Delta t)c_{L_{2c}}(\varepsilon,h)\Delta tKs_h^{n-1,\frac{1}{2}}(u_h^{n'}, u_h^{n'}) + c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n'}\right\|^2_{\Omega_h^{n'}}\\&\quad + c_{L_{2c}}(\varepsilon,h)\Delta tKs_h^{n',\frac{1}{2}}(u_h^n, u_h^n)\end{aligned} \tag{5.46}$$

Substituting (5.46) into (5.44) we have

$$\begin{aligned}&(1-\Delta t\xi_h)\|q_h^n\|^2_{\Omega_h^n}+\frac{\nu}{4}\Delta t\|\nabla q_h^n\|^2_{\Omega_h^n}+\Delta t(2\gamma_s-c_{L_{2c}}(\varepsilon,h)K)s_h^{n',\frac{1}{2}}(u_h^n,u_h^n)\\&+\Delta t(2\gamma_s-(1+c_{L_{2a}}(\varepsilon)\Delta t)c_{L_{2c}}(\varepsilon,h)K)s_h^{n-1,\frac{1}{2}}(u_h^{n'},u_h^{n'})\leq\left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}}\end{aligned}\tag{5.47}$$

with

$$(1-\Delta t\xi_h)\|q_h^n\|^2_{\Omega_h^n}:=(1-\Delta t\xi_h)\|u_h^n\|^2_{\Omega_h^n}+(1-\Delta t\xi_h-(1+c_{L_{2a}}(\varepsilon)\Delta t)^2)\left\|u_h^{n'}\right\|^2_{\Omega_h^{n-1}}\tag{5.48a}$$

$$\frac{\nu}{4}\Delta t\|\nabla q_h^n\|^2_{\Omega_h^n}:=\frac{\nu}{4}\Delta t\|\nabla u_h^n\|^2_{\Omega_h^n}+\nu\Delta t(\frac{1}{4}-(1+c_{L_{2a}}(\varepsilon))c_{L_{2b}}(\varepsilon))\left\|\nabla u_h^{n'}\right\|^2_{\Omega_h^{n-1}}-c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n'}\right\|^2_{\Omega_h^{n'}}\tag{5.48b}$$

We choose $\varepsilon\leq\frac{\nu}{2c_{L_2}c_{\delta_h}\mathbf{w}^\mathbf{n}_\infty}$ so that $c_{L_{2b}}(\varepsilon)\leq\frac{1}{2}$ and $c_{L_{2a}}(\varepsilon)$ and $c_{L_{2c}}(\varepsilon,h)$ are bounded independent of h and Δt. Assume $\gamma_s\geq c_{L_{2c}}(\varepsilon,h)K$ and summe up over $n=1,2,...,k,\ k\leq N$ yields

$$\begin{aligned}&(1-\Delta t\xi_h)\left\|u_h^k\right\|^2_{\Omega_h^k}+\frac{\nu}{4}\Delta t\sum_{n=1}^k\|\nabla u_h^n\|^2_{\Omega_h^n}+\gamma_s\Delta t\sum_{n=1}^k(s_h^{n',\frac{1}{2}}(u_h^n,u_h^n)+k_1s_h^{n-1,\frac{1}{2}}(u_h^{n'},u_h^{n'}))\\&\leq\left\|u_h^0\right\|^2_{\Omega_h^0}+\xi_h\Delta t\sum_{n=1}^k\left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}}+\gamma_s\Delta tk_1s_h^{0,\frac{1}{2}}(u_h^{\frac{1}{2}},u_h^{\frac{1}{2}})\end{aligned}\tag{5.49}$$

where $k_1=1+c_{L_{2a}}(\varepsilon)\Delta t$. Applying Grownwall's Lemma with $\Delta t\xi_h\leq\frac{1}{4}$ we obtain the result with $c_{T_1}:=\xi_h\Delta t$. □

Stability analysis of second characterization

Theorem 4. *Under Assumption 1 we assume u_h^0 well-defended in Ω_h^0, $\frac{1}{2\xi_h}\leq\Delta t\leq C_\nu h^2$ and $\nu\leq1$, the numerical solution of (4.22) satisfies the following estimate*

$$\left\|u_h^k\right\|^2_{\Omega_h^k}+\Delta t(\frac{\nu}{4}\sum_{n=1}^k\|\nabla u_h^n\|^2_{\Omega_h^n}+\gamma_s\sum_{n=1}^k(s_h^{n-1,2}(u_h^{n,e},u_h^{n,e})+s_h^{n,1}(u_h^{n,i},u_h^{n,i})))\leq\exp(c_{T_1}t_k)|||u_h^0|||_0\tag{5.50}$$

with c_{T_1} independent of h and Δt and $|||\cdot|||_0$ defined as

$$|||v|||_0:=(\left\|v^0\right\|^2_{\Omega^0}+\frac{1}{4}\Delta t\nu\left\|\nabla v^0\right\|^2_{\Omega^0}+\frac{1}{2}\gamma_s\Delta ts^0(v^0,v^0))^{\frac{1}{2}}\tag{5.51}$$

Proof. We test the first equation of (4.22) with $2u_h^{n,e}$ and multiply wiht Δt we obtain

$$\|u_h^{n,e}\|^2_{\Omega_h^{n-1}} + \left\|u_h^{n,e} - u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} + 2\Delta t a_h^{n-1}(u_h^{n-1}, u_h^{n,e}) + 4\Delta t \gamma_s s_h^{n-1,2}(u_h^{n,e}, u_h^{n,e}) = \left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} \tag{5.52}$$

Then we test the second equation of (4.22) with $2u_h^{n,i}$ and multiply with Δt we obtain

$$\left\|u_h^{n,i}\right\|^2_{\Omega_h^n} + \left\|u_h^{n,i} - u_h^{n-1}\right\|^2_{\Omega_h^n} + 2\Delta t a_h^n(u_h^{n,i}, u_h^{n,i}) + 4\Delta t \gamma_s s_h^{n,1}(u_h^{n,i}, u_h^{n,i}) = \left\|u_h^{n-1}\right\|^2_{\Omega_h^n} \tag{5.53}$$

Combine (5.52) and (5.53) we have

$$\begin{aligned}
&\|u_h^{n,e}\|^2_{\Omega_h^{n-1}} + \left\|u_h^{n,i}\right\|^2_{\Omega_h^n} + \left\|u_h^{n,e} - u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} + \left\|u_h^{n,i} - u_h^{n-1}\right\|^2_{\Omega_h^n} \\
&+ 2\Delta t(a_h^{n-1}(u_h^{n-1}, u_h^{n,e}) + a_h^n(u_h^{n,i}, u_h^{n,i})) + 4\Delta t\gamma_s(s_h^{n-1,2}(u_h^{n,e}, u_h^{n,e}) + s_h^{n,1}(u_h^{n,i}, u_h^{n,i})) \\
&= \left\|u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} + \left\|u_h^{n-1}\right\|^2_{\Omega_h^n}
\end{aligned} \tag{5.54}$$

Using the lower bound on $a^n(\cdot,\cdot)$ we have

$$a_h^n(u_h^{n,i}, u_h^{n,i}) \geq \frac{\nu}{2}\left\|\nabla u_h^{n,i}\right\|^2_{\Omega_h^n} - \xi_h \left\|u_h^{n,i}\right\|^2_{\Omega_h^n}. \tag{5.55}$$

Using the lower bound on $a^{n-1}(\cdot,\cdot)$ we have

$$\begin{aligned}
&a_h^{n-1}(u_h^{n-1}, u_h^{n,e}) = a_h^{n-1}(u_h^{n,e}, u_h^{n,e}) + a_h^{n-1}(u_h^{n-1} - u_h^{n,e}, u_h^{n,e}) \\
&\geq \frac{\nu}{2}\|\nabla u_h^{n,e}\|^2_{\Omega_h^{n-1}} - \xi_h\|u_h^{n,e}\|^2_{\Omega_h^{n-1}} - \left\|a_h^{n-1}\right\|\left\|u_h^{n-1} - u_h^{n,e}\right\|_{\Omega_h^{n-1}}\|\nabla u_h^{n,e}\|_{\Omega_h^{n-1}} \\
&\geq \frac{\nu}{2}\|\nabla u_h^{n,e}\|^2_{\Omega_h^{n-1}} - \xi_h\|u_h^{n,e}\|^2_{\Omega_h^{n-1}} - c_h\left\|u_h^{n,e} - u_h^{n-1}\right\|_{\Omega_h^{n-1}}\|\nabla u_h^{n,e}\|_{\Omega_h^{n-1}} \\
&\geq \frac{\nu}{2}\|\nabla u_h^{n,e}\|^2_{\Omega_h^{n-1}} - \xi_h\|u_h^{n,e}\|^2_{\Omega_h^{n-1}} - \frac{2C^2h^{-2}}{\nu}\left\|u_h^{n,e} - u_h^{n-1}\right\|^2_{\Omega_h^{n-1}} - \frac{\nu}{4}\|\nabla u_h^{n,e}\|^2_{\Omega_h^{n-1}} \\
&\geq \frac{\nu}{4}\|\nabla u_h^{n,e}\|^2_{\Omega_h^{n-1}} - \xi_h\|u_h^{n,e}\|^2_{\Omega_h^{n-1}} - \frac{2C^2h^{-2}}{\nu}\left\|u_h^{n,e} - u_h^{n-1}\right\|^2_{\Omega_h^{n-1}}
\end{aligned} \tag{5.56}$$

with $c_h := \left\|a_h^{n-1}\right\| = \sup_{v_h, w_h \in V_h^{n-1}} \frac{a_h^{n-1}(v_h, \nabla w_h)}{\|v_h\|_{L^2}\|\nabla w_h\|_{L^2}} \leq Ch^{-1}$.

Substituting (5.55) and (5.56) into (5.54) we have

$$
\begin{aligned}
&(1-2\xi_h\Delta t)(\left\|u_h^{n,i}\right\|_{\Omega_h^n}^2+\|u_h^{n,e}\|_{\Omega_h^{n-1}}^2)+\nu\Delta t(\left\|\nabla u_h^{n,i}\right\|_{\Omega_h^n}^2+\frac{1}{2}\|\nabla u_h^{n,e}\|_{\Omega_h^{n-1}}^2)+\left\|u_h^{n,i}-u_h^{n-1}\right\|_{\Omega_h^n}^2\\
&+(1-\frac{4C^2h^{-2}}{\nu}\Delta t)\left\|u_h^{n,e}-u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2+4\Delta t\gamma_s(s_h^{n-1,2}(u_h^{n,e},u_h^{n,e})+s_h^{n,1}(u_h^{n,i},u_h^{n,i}))\\
&\leq\left\|u_h^{n-1}\right\|_{\Omega_h^n}^2+\left\|u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2
\end{aligned}
\tag{5.57}
$$

Let $1-\frac{4C^2h^{-2}}{\nu}\Delta t\geq 0$ i.e. $\Delta t\leq\frac{\nu}{4C^2h^{-2}}=2C_\nu h^2$ we have

$$(1-\frac{4C^2h^{-2}}{\nu}\Delta t)\left\|u_h^{n,e}-u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2\geq 0. \tag{5.58}$$

Since $\left\|u_h^{n,i}-u_h^{n-1}\right\|_{\Omega_h^n}^2\geq 0$, (5.57) can be written as

$$
\begin{aligned}
&(1-2\Delta t\xi_h)(\left\|u_h^{n,i}\right\|_{\Omega_h^n}^2+\|u_h^{n,e}\|_{\Omega_h^{n-1}}^2)+\frac{\nu}{2}\Delta t(2\left\|\nabla u_h^{n,i}\right\|_{\Omega_h^n}^2+\|\nabla u_h^{n,e}\|_{\Omega_h^{n-1}}^2)\\
&+4\Delta t\gamma_s(s_h^{n-1,2}(u_h^{n,e},u_h^{n,e})+s_h^{n,1}(u_h^{n,i},u_h^{n,i}))\leq\left\|u_h^{n-1}\right\|_{\Omega_h^n}^2+\left\|u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2
\end{aligned}
\tag{5.59}
$$

Since $u_h^n=\frac{u_h^{n,i}+u_h^{n,e}}{2}$,

$$2\|u_h^n\|_{\Omega_h^n}^2\leq\left\|u_h^{n,i}\right\|_{\Omega_h^n}^2+\|u_h^{n,e}\|_{\Omega_h^n}^2 \tag{5.60a}$$

$$2\|\nabla u_h^n\|_{\Omega_h^n}^2\leq\left\|\nabla u_h^{n,i}\right\|_{\Omega_h^n}^2+\|\nabla u_h^{n,e}\|_{\Omega_h^n}^2 \tag{5.60b}$$

(5.59) can be written as

$$
\begin{aligned}
&(1-2\xi_h\Delta t)2\|u_h^n\|_{\Omega_h^n}^2+\nu\Delta t\|\nabla u_h^n\|_{\Omega_h^n}^2+4\Delta t\gamma_s(s_h^{n-1,2}(u_h^{n,e},u_h^{n,e})+s_h^{n,1}(u_h^{n,i},u_h^{n,i}))\\
&\leq\left\|u_h^{n-1}\right\|_{\Omega_h^n}^2+\left\|u_h^{n-1}\right\|_{\Omega^{n-1}}^2
\end{aligned}
\tag{5.61}
$$

Applying Lemma (2) to (5.61) we obtain

$$
\begin{aligned}
&(1-2\xi_h\Delta t)\|u_h^n\|_{\Omega_h^n}^2+\frac{1}{2}\nu\Delta t\|\nabla u_h^n\|_{\Omega_h^n}^2+2\Delta t\gamma_s(s_h^{n-1,2}(u_h^{n,e},u_h^{n,e})+s_h^{n,1}(u_h^{n,i},u_h^{n,i}))\\
&\leq\frac{1}{2}(\left\|u_h^{n-1}\right\|_{\Omega_h^n}^2+\left\|u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2)\leq\frac{1}{2}(\left\|u_h^{n-1}\right\|_{\mathcal{O}_{\delta_h}(\Omega_h^{n-1})}^2+\left\|u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2)\\
&\leq(1+\frac{c_{L_{2a}}(\varepsilon)\Delta t}{2})\left\|u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2+\frac{1}{2}c_{L_{2b}}(\varepsilon)\Delta t\nu\left\|\nabla u_h^{n-1}\right\|_{\Omega_h^{n-1}}^2\\
&+\frac{1}{2}c_{L_{2c}}(\varepsilon,h)\Delta tKs_h^{n-1}(u_h^{n-1},u_h^{n-1})
\end{aligned}
\tag{5.62}
$$

We choose $\varepsilon \leq \frac{\nu}{2c_{L_2} c_{\delta_h} \mathbf{w}^{\mathbf{n}}_{\infty}}$ so that $c_{L_{2b}}(\varepsilon) \leq \frac{1}{2}$ and $c_{L_{2a}}(\varepsilon)$ and $c_{L_{2c}}(\varepsilon, h)$ are bounded independent of h and Δt. We assume $\gamma_s \geq c_{L_{2c}} K$ and summ up over $n = 1, 2, ..., k,\ k \leq N$ yields

$$\begin{aligned}
&\left\| u_h^k \right\|^2_{\Omega_h^k} + \frac{1}{4}\nu \Delta t \sum_{n=1}^{k} \|\nabla u_h^n\|^2_{\Omega_h^n} + \Delta t \gamma_s \sum_{n=1}^{k} (s_h^{n-1,2}(u_h^{n,e}, u_h^{n,e}) + s_h^{n,1}(u_h^{n,i}, u_h^{n,i})) \\
&\leq \left\| u_h^0 \right\|^2_{\Omega_h^0} + (\frac{c_{L_{2a}}(\varepsilon)}{2} + 2\xi_h) \Delta t \sum_{n=0}^{k-1} \|u_h^n\|^2_{\Omega_h^n} + \frac{1}{4}\Delta t \nu \left\| \nabla u_h^0 \right\|^2_{\Omega_h^0} + \frac{1}{2}\gamma_s \Delta t s_h^0(u_h^0, u_h^0)
\end{aligned} \tag{5.63}$$

We apply Gronwall's Lemma with $\xi_h \Delta t \leq \frac{1}{4}$ and obtain the result with $c_{T_1} = \frac{c_{L_{2a}}(\varepsilon)}{2} + 2\xi_h$. □

Chapter 6

Numerical Experiments

In this section, we present numerical experiments for higher order stabilized time stepping mehtods proposed and analyzed in the last chapters.

6.1 Setup

In all numerical experiment we use a bilinear form which is slightly different with the bilinear form used in the analysis section, it is given as

$$a_h^n(u_h, v_h) = \int_{\Omega_h^n} \nu \nabla u_h \cdot \nabla v_h dx + \int_{\Omega_h^n} (\mathbf{w}^e \cdot \nabla u_h) v_h + \int_{\Omega_h^n} \text{div}(\mathbf{w}^e) u_h v_h dx \tag{6.1}$$

We also use a right hand side source term

$$f(v_h) := \int_{\Omega_h^n} f v_h dx. \tag{6.2}$$

All the implementations are done in *ngsxfem* [15] which is an Add-on package for unfitted FEM in the general purpose finite element solver *Netgen/NGSolve*.

In numerical experiments, the discrete domains Ω_h^n are approximated by an interpolation ϕ_h^n of a level set $\phi(t_n)$ i.e. $\Omega_h^n := \{\phi_h^n < 0\}$. The strip domains $\mathcal{S}_{l\cdot\delta_h}^{\pm}$ can be realized by a level set $\{|\phi_h^n| \leq l \cdot \delta_h\}$ which is a approximate signed distance function.

6.2 Example: Traveling circle

We consider a circle

$$\Omega_t = \{\mathbf{x} \in \mathbb{R}^2 : \phi(\mathbf{x}, t) < 0\} \tag{6.3}$$

as moving domain traveling with a time-dependent velocity field that is constant in space through a background mesh, where

$$\begin{aligned} \phi(\mathbf{x},t) &= \|\mathbf{x} - \rho(\mathbf{x},t)\| - R_0 \\ \rho(\mathbf{x},t) &= (\frac{1}{\pi}\sin(2\pi t), 0)^T \\ \mathbf{w}(\mathbf{x},t) &= \partial_t \rho(\mathbf{x},t) \end{aligned} \tag{6.4}$$

with $R_0 = 0.5$. Fixing the background domain $\tilde{\Omega} = (-1.2, 1.2) \times (-1, 1)$ we consider the time interval $[0, T]$ with $T = 0.5$ for implicit stabilized time stepping mehtods and $T = 0.1$ for explcit stabilized time stepping method.

For the simplicity, we set the parameter $\nu = 1$ and the extended solution is given as

$$u^e(\mathbf{x},t) = \cos^2(\frac{\pi}{2R_0}\|\mathbf{x} - \rho(\mathbf{x},t)\|_2) \tag{6.5}$$

which fullfills the flux boundary condition given in chapter 2. We set mesh size $h = 0.2$ and $\Delta t \in \{\frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}\}$ for stabilized implicit Euler time stepping method, stabilized BDF2 time stepping method and stabilized BDF3 time stepping method. Due to Theorem 2, we have $\Delta t \leq C_\nu h^2$, therefore we set mesh size $h = 0.2$ and $\Delta t \in \{\frac{1}{256}, \frac{1}{512}, \frac{1}{1024}, \frac{1}{2048}, \frac{1}{4096}, \frac{1}{8192}, \frac{1}{16384}\}$ for stablized explicit Euler time stepping method. We compute asymptotical convergence rates in space and time by "experimental order of convergence" as

$$order = \frac{\text{Log}(error_1) - \text{Log}(error_2)}{\text{Log}(\Delta h_1) - \text{Log}(\Delta h_2)} \tag{6.6}$$

6.2.1 Convergence in space and time

We consider the errors in the discrete space-time errors

$$\|u_h - u^{\mathrm{e}}\|_{L^2(L^2)} := \sum_{n=1}^{N} \Delta t \|u_h - u^{\mathrm{e}}\|^2_{\Omega_h^n}. \tag{6.7}$$

The results are given in the form of Tables. The columns represent the different levels of space refinements. The rows represent the different levels of time refinements. The last column shows the experimental order of convergence with respect to time refinements on the fines spacial mesh. Similarly, the second last row shows the experimental order of convergence in space with respect to the smallest time step. The last row for stabilized implicit Euler time stepping method shows the experimental order of convergence with respect to space refinement on each level and time refinement on each two levels. The last row for stabilized BDF2 and BDF3 time stepping methods shows the experimental order of convergence with respect to both time and space refinements on each level. We also add the eoc of combined refinement on each space level and 3 levels of

$L_t \downarrow\ L_x \rightarrow$	0	1	2	eoc$_t$
0	2.2×10^{-2}	1.6×10^{-2}	1.6×10^{-2}	-
1	2.0×10^{-2}	9.2×10^{-3}	8.3×10^{-3}	0.95
2	1.9×10^{-2}	6.2×10^{-3}	4.5×10^{-3}	0.92
3	1.9×10^{-2}	5.2×10^{-3}	2.5×10^{-3}	0.90
4	1.9×10^{-2}	4.9×10^{-3}	1.7×10^{-3}	0.81
eoc$_x$	-	1.96	1.53	
eoc$_{xtt}$	-	0.91	0.93	

Table 6.1: $L^2(L^2)$ error for stabilized implicit Euler time stepping method

$L_t \downarrow\ L_x \rightarrow$	0	1	2	3	eoc$_t$
0	1.4×10^{-2}	7.0×10^{-3}	6.3×10^{-3}	6.2×10^{-3}	-
1	1.3×10^{-2}	4.2×10^{-3}	2.5×10^{-3}	2.2×10^{-3}	1.49
2	1.3×10^{-2}	3.4×10^{-3}	1.2×10^{-3}	7.6×10^{-4}	1.53
3	1.3×10^{-2}	3.2×10^{-3}	8.9×10^{-4}	3.2×10^{-4}	1.25
eoc$_x$	-	2.02	1.85	1.48	
eoc$_{xt}$	-	1.74	1.80	1.91	

Table 6.2: $L^2(L^2)$ error for BDF2 stabilized time stepping method

refinements in time eoc$_{xttt}$ for stabilized explicit Euler time stepping method.

Results of implicit methods

I. In Table 6.1 we display the $L^2(L^2)$ norm for the 5 different time levels and 3 different space levels and corresponding eocs for stabilized implicit Euler time stepping method. We observe that the convergence $\|u_h - u^{\mathrm{e}}\|_{L^2(L^2)} \leq h^2 + \Delta t$.

II. In Table 6.2 we display the $L^2(L^2)$ norm for four different time and space levels and corresponding eocs for stabilized BDF2 time stepping method. We observe that the convergence $\|u_h - u^{\mathrm{e}}\|_{L^2(L^2)} \leq h^2 + \Delta t^2$ and there is an improved rate in time.

III. In Table 6.3 we display the $L^2(L^2)$ norm for the four different time and space levels and corresponding eocs for stabilized BDF3 time stepping method. We observe that the convergence $\|u_h - u^{\mathrm{e}}\|_{L^2(L^2)} \leq h^2 + \Delta t^3$.

Results of explicit methods

In Table 6.4 we display the the $L^2(L^2)$ norm for the 7 different time levels and 3 different space levels and corresponding eocs for stabilized explicit Euler time stepping method. We observe that the method tends to be stable after refining time on 6 levels and space on 2 levels. We find that the convergence behavior in space is better than the convergence behavior in time. We expect that the convergence can be $\|u_h - u^{\mathrm{e}}\|_{L^2(L^2)} \leq h^2 + \Delta t$ after more refinements on time and space, but it may lead to much cost on computation.

$L_t \downarrow$ $L_x \rightarrow$	0	1	2	3	eoc$_t$
0	2.4×10^{-2}	1.3×10^{-2}	1.1×10^{-2}	1.0×10^{-2}	-
1	2.1×10^{-2}	6.4×10^{-3}	3.4×10^{-3}	2.7×10^{-3}	1.89
2	2.0×10^{-2}	5.0×10^{-3}	1.5×10^{-3}	6.0×10^{-4}	2.17
3	2.0×10^{-2}	4.9×10^{-3}	1.2×10^{-3}	3.3×10^{-4}	0.86
eoc$_x$	-	2.03	2.03	1.86	
eoc$_{xt}$	-	1.91	2.09	2.18	

Table 6.3: $L^2(L^2)$ error for stabilized BDF3 time stepping method

$L_t \downarrow$ $L_x \rightarrow$	0	1	2	eoc$_t$
0	1.0×10^{-1}	-	-	-
1	1.2×10^{-1}	-	-	-
2	1.5×10^{-1}	-	-	-
3	1.7×10^{-1}	3.3×10^{-2}	-	-
4	2.0×10^{-1}	4.0×10^{-2}	-	-
5	2.1×10^{-1}	4.6×10^{-2}	6.8×10^{-3}	-
6	2.1×10^{-1}	5.1×10^{-2}	8.1×10^{-3}	-
eoc$_x$	-	2.04	2.65	
eoc$_{xttt}$	-	1.60	2.03	

Table 6.4: $L^2(L^2)$ error for stabilized explicit Euler time stepping method

Chapter 7

Conclusion

In this thesis we have introduced some higher order unfitted finite element methods for solving PDEs posed on moving domains. The numerical methods are based on a standard geometrically unfitted finite element with a stabilization term for space discretization and higher order time discretizations such as BDF2, BDF3 methods and tripezoidal rules. In the analysis, We give a full stability analysis on stabilized BDF2 and explicit time stepping mehtods. We also give a reasonable argument on the stability analysis on tripezoidal rules for moving domains.

For the implementation, we were able to derive optimal order error bounds in the L_2 norm for BDF2 and BDF3 stabilized time stepping methods. We also presented a interesting result of stabilized explicit Euler time stepping method.

Open Problems

I. In the numerical experiments we only implement BDF2, BDf3 and explicit stabilized time stepping methods. We are intereted in implementing tripezoidal rules and want to expect to obatin an interesting and reasonable result.

II. We only consider a simple model problem, travelling circle. Many applications involve more complex problems, e.g. random travelling object with random change of shape and two-phase Navier-Stokes equations,etc. An extension of this method to those complex problems will be challenging and interesting, it should be investigated and analyzed in the future.

III. The numerical example we have treated is only in 2D simple model, hence it is a clearly challenge to implement a 3D simple model and extend it to a 3D comlex model. It will be an interesting task and worth pursuing.

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