Unfitted mixed finite element methods

Guosheng Fu[1], Christoph Lehrenfeld[2], Tim van Beeck[2]

[1] UNIVERSITY OF NOTRE DAME [2] GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

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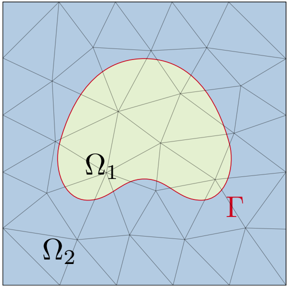
$$\begin{pmatrix} A_{\text{?👻?}} & B^T_{\text{?👻?}} \\ B_{\text{?👻?}} & -\text{?👻?} \end{pmatrix} \cdot \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

Background: Unfitted FEM

Problems

- PDEs on domains with separate geometry description (e.g. level set)
- PDEs on embedded surfaces
- PDEs on moving domains



Challenges

- FE formulation in unfitted setting
- Stability/robustness for arbitrary (small) cuts
- Imposition of boundary/interface conditions
- Cut integration (robust / high order accurate)

Solution techniques

- unfitted FE spaces (CutFEM / XFEM / Finite Cell / Unfitted FEM / ...)
- Weak imposition of boundary conditions through Nitsche / stab. Lagrange mult.
- Ghost (👻) penalty / aggregated FEM

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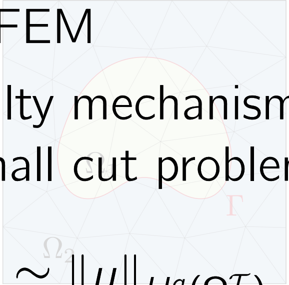

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Assumptions:

- we want to do unfitted FEM
- we know the ghost penalty mechanism as one tool to deal with small cut problems:

$$\|u\|_{H^q(\Omega)} + \|u\|_{👻,q} \simeq \|u\|_{H^q(\Omega^{\mathcal{T}})}$$

- geometry handling is exact

Examples: Mixed formulation of the Poisson and Stokes problems

Mixed Poisson/Darcy:
Find u, p with $p = p_D$ on $\partial\Omega$, s.t.

$$\begin{aligned} K^{-1}u - \nabla p &= 0 \quad \text{in } \Omega, \\ \operatorname{div} u &= -f \quad \text{in } \Omega. \end{aligned}$$

Stokes:
Find u, p with $u = u_D$ on $\partial\Omega$, s.t.

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \end{aligned}$$

Constraint equation correspond to mass conservation (p is Lagrange multiplier).

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General unfitted saddle point problems

Find $(u, p) \in \Sigma \times Q$, s.t.

$$\begin{aligned} a(u, v) + b(v, p) &= g(v), \quad \forall v \in \Sigma, \\ b(u, q) &= h(q), \quad \forall q \in Q. \end{aligned}$$

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ϟ (inf-sup) stability in the presence of arbitrary cuts ?

Literature I/II: 👻-penalties on u and p:

[neglecting bound. conditions]

- Stokes (advantage: H1-conformity) based on stable fitted method:
  - Stabilized vel./press. pairs[1]
  - Taylor-Hood [2,3]
  - Scott-Vogelius (macro-element version, exactly divfree*) [+grad-div)][4]
- Poisson/Darcy and Stokes-Darcy based on stable fitted method:
  - $\mathbb{RT}^k$ / $\mathbb{BDM}^k \times \mathbb{P}^k$ (inf-sup-stable (in the fitted case) pairs)[5,6]

[1] A. Massing, M.G. Larson, A. Logg, M.E. Rognes, *A stabilized Nitsche fictitious domain method for the Stokes problem*. J. Sc. Comp., 2014

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Stabilized formulations:

$$\text{Find } (u_h, p_h) \in \Sigma \times Q, \text{ s.t.} \quad \begin{array}{rcrcll} (a_{(h)} + \text{👻}_u)(u_h, v_h) & + & b_{(h)}(v_h, p_h) & = & g(v_h), & \forall v_h \in \Sigma_h, \\ b_{(h)}(u_h, q_h) & - & (d_h + \text{👻}_p)(p_h, q_h) & = & h(q_h), & \forall q_h \in Q_h. \end{array}$$

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$\rightsquigarrow$ inf-sup-stability of global bilinear form (independent of cut position) 👌, but mass conservation polluted ⚡

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Literature II/II: No 👻-penalty on p-q-coupling [neglecting bound. conditions]

For Darcy interface problem:

- $\mathbb{RT}^0 \times \mathbb{P}^0$ (inf-sup-stable pair*) (low order, 2D) [7]
- $\mathbb{RT}/\mathbb{BDM} \times \mathbb{P}^k$, inf-sup-stable pair* + 👻-penalties for divergence [8]

$$\text{Find } (u_h, p_h) \in \Sigma_h \times Q_h \text{, s.t.} \quad \begin{array}{lcl} (a + 👻_u)(u_h, v_h) \; + (b + 👻^*)(v_h, p_h) & = & g(v_h), \quad \forall v_h \in \Sigma_h, \\ (b + 👻^*)(u_h, q_h) & = & h(q_h), \quad \forall q_h \in Q_h. \end{array}$$

$$👻^*(v_h, q_h) = 👻_p(\operatorname{div} v_h, q_h) = \sum_{F \in \mathcal{F}_h} \sum_{j=0}^{k} \gamma h^{2j+1} \int_F [\![D^j \operatorname{div} v_h]\!] [\![D^j q_h]\!] ds$$

- mass balance hardly polluted
  (effectively using a smooth ext of f on the active mesh, possibly by patch-👻-penalties)
- $f \in \mathbb{P}^k \Rightarrow \operatorname{div} u_h = -f$

[7] C.D'Angelo, A. Scotti, *A mixed finite element method for Darcy flow in fractured porous media with non-matching grids*. ESAIM:M2AN, 2012

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Aim now: Robustness w.r.t. cut position (also high order) w/o pollution of mass balance

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A step back: Fitted mixed Poisson (recap)

Find $u_h \in \Sigma_h = \mathbb{RT}^k \subset H(\mathrm{div}, \Omega)$, $p_h \in Q_h = \mathrm{div}\, \Sigma_h = \mathbb{P}^k \subset L^2(\Omega)$, s.t.

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\begin{aligned}
(u_h, v_h)_\Omega + \quad (\mathrm{div} v_h, p_h)_\Omega &= \quad g(v_h) = (v_h \cdot n, p_D)_{\partial\Omega} \forall\ q_h \in \Sigma_h, \\
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- $\Sigma_h = \Sigma_h^0 \oplus_a \Sigma_h^\perp$ with $\Sigma_h^0 = \ker b = \{u_h \in \Sigma_h \mid \text{div}\, u_h = 0\}$ and $a(\cdot, \cdot) = (\cdot, \cdot)_\Omega$.
- 3 subproblems for 3 unknowns: $(u_h, p_h) \rightsquigarrow (u_h^0, u_h^\perp, p_h)$
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- Discrete LBB-stability: $\inf_{q_h} \sup_{u_h} \dfrac{b(u_h, q_h)}{\|u_h\|_\Sigma \|q_h\|_Q} \geq c > 0 \quad \Rightarrow$ stability of (2) & (3)

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- With $\mathrm{div}\, \Sigma_h \subset Q_h$ (2) also reads as $\mathrm{div}\, u_h + \Pi_{Q_h} f = 0 \quad$ (*pointwise*)

Unfitted Mixed FEM

Starting point: straight-forward unfitted Mixed FEM:

Find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h) \subset H(\mathrm{div}, \Omega^{\mathcal{T}})$, $p_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h) \subset L^2(\Omega^{\mathcal{T}})$, s.t.

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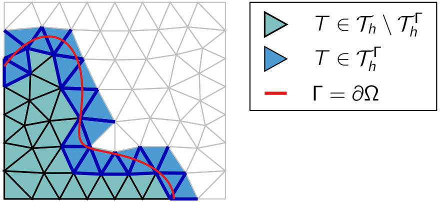

(1) Determine u_h^0 from $a_h(u_h^0, v_h^0) = (u_h^0, v_h^0)_\Omega = g(v_h^0)\ \forall v_h^0 \in \Sigma_h^0$

(2) Determine $u_h^\perp$ from $b_h(u_h^\perp, q_h) = (\mathrm{div}\ u_h^\perp, q_h)_\Omega = h(q_h)\ \forall q_h \in Q_h$

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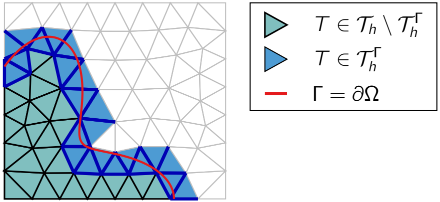
Adjusted unfitted mixed FEM



Observation on the subspace Σ_h^0

Due to $\operatorname{div} \Sigma_h \subset Q_h$ we have $\ker b =$ *pointwise divergence-free* functions

$$\Rightarrow \Sigma_h^0 = \{u_h \in \Sigma_h \mid b(u_h, q_h) = 0 \ \forall q_h \in Q_h\} = \ker b = \ker b_h \text{ with } b_h(u_h, q_h) := (\operatorname{div} u_h, q_h)_{\Omega^{\mathcal{T}}}$$

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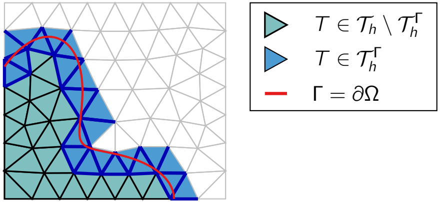

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Find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h)$, $\bar{p}_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h)$, s.t.

$$\begin{aligned} (u_h, v_h)_\Omega + \gamma_{\text{👻}} \text{👻}(u_h, v_h) + (\operatorname{div} v_h, \bar{p}_h)_{\Omega^\mathcal{T}} &= g(v_h) = (v_h \cdot n, p_D)_{\partial\Omega} && \forall\, v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_{\Omega^\mathcal{T}} &= h_h(q_h) = (-f_h, q_h)_{\Omega^\mathcal{T}} && \forall\, q_h \in Q_h. \end{aligned}$$

- $\gamma_{\text{👻}} > 0$ ($\gamma_{\text{👻}} = 0$ possible)
- Assume $f_h \in Q_h$ with $f_h \approx \mathcal{E} f$ in $\Omega^\mathcal{T}$ (with $\mathcal{E}$ smooth ext. op. from Ω to $\Omega^\mathcal{E} \supset \Omega^\mathcal{T}$.)

The subproblems for u_h:

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Symmetric saddle point problem; well-conditioned linear systems.

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Symmetric saddle point problem; well-conditioned linear systems.

Subproblems (1) & (2) for $u_h^0 \in \Sigma_h^0$ and $u_h^\perp \in \Sigma_h^\perp$:

- $a_h(u_h^0, v_h^0) = (u_h^0, v_h^0)_\Omega + \gamma_{👻} 👻(u_h^0, v_h^0) = g(v_h^0) = (v_h^0, p_D)_{\partial\Omega}\ \forall v_h^0 \in \Sigma_h^0$
  Consistent, continuous, coercive (w.r.t. $\|\cdot\|_{H(\operatorname{div};\Omega^\mathcal{T})}$).

[8] T. Frachon, P. Hansbo, E. Nilsson, S. Zahedi, *A divergence preserving cut finite element method for Darcy flow.* arXiv: 2205.12023

The subproblems for u_h:

Adjusted unfitted Mixed FEM:

Find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h)$, $\bar{p}_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h)$, s.t.

$$\begin{aligned} (u_h, v_h)_\Omega + \gamma_{👻} 👻(u_h, v_h) + (\operatorname{div} v_h, \bar{p}_h)_{\Omega^\mathcal{T}} &= g(v_h) = (v_h \cdot n, p_D)_{\partial\Omega} \quad \forall\, v_h \in \Sigma_h, \\ (\operatorname{div} u_h, q_h)_{\Omega^\mathcal{T}} &= h_h(q_h) = (-f_h, q_h)_{\Omega^\mathcal{T}} \quad \forall\, q_h \in Q_h. \end{aligned}$$

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Error estimate for u_h

[u_h is the same as in [8] if f_h is a 👻-penalty-based discrete ext. of f]

$$\|u - u_h\|_{H(\operatorname{div};\Omega^{\mathcal{T}})} \lesssim \|u - \Pi^{\Sigma_h} u\|_{L^2(\Omega^{\mathcal{T}})} + \|\Pi^{Q_h}\mathcal{E}f - f_h\|_{L^2(\Omega^{\mathcal{T}})} \lesssim h^{k+1},$$

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Lagrange Multiplier $\bar{p}_h$

Subproblem (3) for $\bar{p}_h \in Q_h$:

$$b_h(v_h^\perp, \bar{p}_h) = (\operatorname{div} v_h^\perp, \bar{p}_h)_{\Omega^\mathcal{T}} = g(v_h^\perp) - (u_h^\perp, v_h^\perp)_{\Omega^\mathcal{T}} = (v_h^\perp \cdot n, p_D)_{\partial\Omega} - (u_h^\perp, v_h^\perp)_{\Omega^\mathcal{T}} \;\forall v_h^\perp \in \Sigma_h^\perp$$

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- inconsistent on cut elements, i.e. $\bar{p}_h \not\approx p$ (part. integration "does not work")
- consistent on uncut elements

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- $\rightsquigarrow$Replace (3) with a different way to obtain p_h
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Element-local post-processing:

On each element $T \in \mathcal{T}_h$:

$$(\nabla p_h^\star, \nabla q_h^\star)_T = (u_h, \nabla q_h^\star)_T \qquad \forall q_h^\star \in \mathcal{P}^{k+1}(T) \setminus \mathbb{R},$$
$$(p_h^\star, 1)_T = (\bar{p}_h, 1)_T \text{ if } T \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma,$$
$$(p_h^\star, 1)_{T\cap\partial\Omega} = (p_D, 1)_{T\cap\partial\Omega} \text{ if } T \in \mathcal{T}_h^\Gamma.$$

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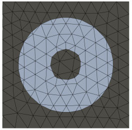
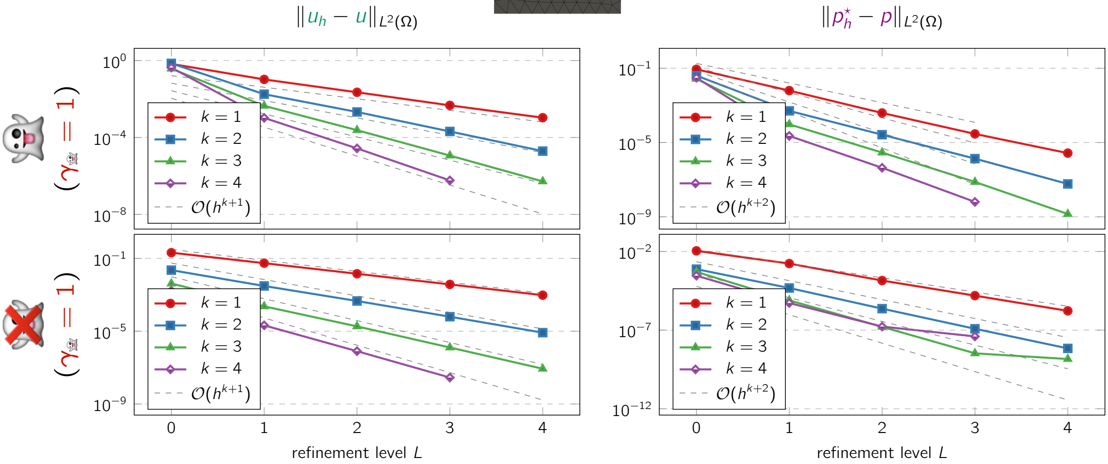
Alternative: Patch-local post-processing (preserve mean value on uncut elements)

Numerical example: mixed Poisson on a ring, manufactured solution

- $\mathbb{RT}^k \times \mathbb{P}^k$
- isoparametric unfitted FEM



- postprocessing involving p_D
- uniform refinements



Neumann boundary conditions: ~~$p = p_D$~~ $\rightsquigarrow u \cdot n = u_{D,n}$ on $\partial\Omega$

Stabilized Lagrange Multiplier Approach

(similar to [9])

Find $u_h \in \Sigma_h = \mathbb{RT}^k(\mathcal{T}_h)$, $\bar{p}_h \in Q_h = \mathbb{P}^k(\mathcal{T}_h)$, $\lambda_h \in \Lambda_h = \mathbb{P}^k(\mathcal{T}_h^{\Gamma})$, s.t.

$$
\begin{aligned}
(u_h, v_h)_\Omega + \gamma_{👻} 👻(u_h, v_h) + \quad (\operatorname{div} v_h, \bar{p}_h)_{\Omega^{\mathcal{T}}} - \quad (v_h \cdot n, \lambda_h)_{\partial\Omega} &= \quad g(v_h) = 0 \forall\ v_h \in \Sigma_h, \\
(\operatorname{div} u_h, q_h)_{\Omega^{\mathcal{T}}} \quad &= \quad (-f_h, q_h)_{\Omega^{\mathcal{T}}} \forall\ q_h \in Q_h, \\
(u_h \cdot n, \mu_h)_{\partial\Omega} \quad - 👻_\lambda(\lambda_h, \mu_h) &= \quad (u_{D,n}, \mu_h)_{\partial\Omega} \forall\ \mu_h \in \Lambda_h.
\end{aligned}
$$

- $👻_\lambda(\cdot,\cdot)$:
  - smoothing type penalties
  - $+$ volume gradient stabilization weakly enforcing $\nabla\lambda_h \cdot n \approx 0$
- $\lambda_h - \mu_h$-block $👻_\lambda(\lambda_h, \mu_h)$ not invertible
- symmetric saddle-point problem
- Mass balance stays "clean" (in the volume).
- Patch-wise postprocessing unaffected.

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[8] T. Frachon, P. Hansbo, E. Nilsson, S. Zahedi, *A divergence preserving cut finite element method for Darcy flow*. arXiv: 2205.12023

[9] E. Burman, *Projection Stabilization of Lagrange Multipliersfor the Imposition of Constraints on Interfacesand Boundaries*. NMPDE, 2013

Conclusion & Outlook

Unfitted mixed FEM

- Circumvent polluting the mass balance (one of the main features of mixed formulations)
- Exploit pointwise character of constraint
  - to go from div-constraint on Ω to $\Omega^{\mathcal{T}}$
  - (or to apply 👻$^{*}(u_h, q_h)$ as in [8])
- Split into 3 subproblems (inconsistency only affects p_h)
- Use post-processing techniques to recover $p_h^{\star}$ (higher order)

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Extensions

- Analysis Neumann case
- $\gamma_{👻} = 0$ possible? (−: condition, postprocess.; +: hybridization) / hybridiz. on patches
- $\rightsquigarrow$ Stokes / Navier-Stokes:
  - velocity space $H(\mathrm{div})$-conforming (H^1-non-conforming) $\rightsquigarrow$ exactly div-free
  - $a_h(\cdot,\cdot)$ includes DG terms and 👻$_u$-penalties
  - $f = f_h = 0$
  - Dirichlet bnd. correspond to Neumann case (last slide)

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Thank you for your attention!

[8] T. Frachon, P. Hansbo, E. Nilsson, S. Zahedi, *A divergence preserving cut finite element method for Darcy flow*. arXiv: 2205.12023