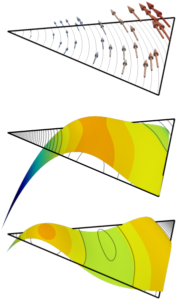
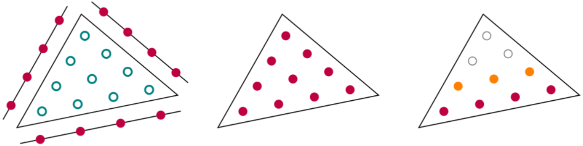
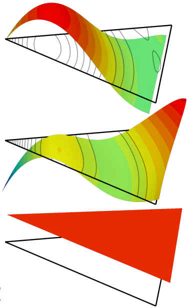
Embedded Trefftz DG methods

or: How to Trefftzify your DG method

Christoph Lehrenfeld, Paul Stocker







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Overview

Repetition of Discontinuous Galerkin (DG) und Hybrid DG methods

Trefftz DG and Embedded Trefftz DG methods

Algorithmic aspects & Numerical examples

Conclusion & Outlook

more details: C.L. & P. Stocker, *Embedded Trefftz DG methods*, `https://arxiv.org/abs/2201.07041`

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DG discretization (for linear PDEs)

$$\mathcal{L}u = f \text{ in } \Omega \subset \mathbb{R}^d + \text{boundary conditions.}$$

A typical standard DG discretization:

$$\text{Find } u_h \in V^p(\mathcal{T}_h), \quad \text{s.t. } a_h(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V^p(\mathcal{T}_h)$$

with polynomial spaces $V^p(K) = \mathcal{P}^p(K)$. Regularity is imposed weakly through $a_h(\cdot, \cdot)$.

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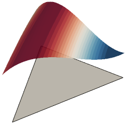
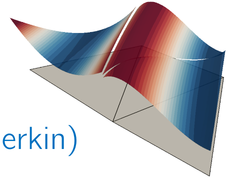




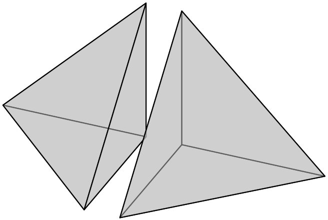
Motivation for DG (instead of continuous Galerkin)

- conservation properties (test function χ_K)
- simple stability mechanism for non-symmetric/non-linear problems (e.g. convection)
- simplicity of data structures / space construction (e.g. polygonal meshes)
- ...

$\rightsquigarrow$ flexibility

Example: Standard DG for Poisson (Symmetric interior penalty DG)

$$\mathcal{L}u = -\Delta u = f \text{ in } \Omega, \quad g = u \text{ on } \partial\Omega.$$



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Find $u_h \in V^p(\mathcal{T}_h)$, s.t. $a_h(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V^p(\mathcal{T}_h)$ with

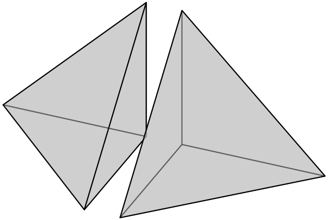
$$a_h(u,v) = \sum_K \int_K \nabla u \nabla v \, dx + \sum_{F \in \mathcal{F}_h^{\text{int}}} \int_F \overbrace{-\{\!\{\partial_n u\}\!\}[\![v]\!]}^{\text{consistency}} \overbrace{-\{\!\{\partial_n v\}\!\}[\![u]\!]}^{\text{symmetry}} \overbrace{+\,\alpha p^2 h^{-1}[\![u]\!][\![v]\!]}^{\text{stability}} \, ds$$

$$+ \sum_{F \in \mathcal{F}_h^{\text{bnd}}} \int_F -\partial_n u \, v - \partial_n v \, u + \alpha p^2 h^{-1} uv \, ds$$

$$\ell(v) = \sum_{F \in \mathcal{F}_h^{\text{bnd}}} \int_F (-\partial_n v + \alpha p^2 h^{-1} v) g \, ds.$$

$\{\!\{\cdot\}\!\}$: average across facets, $[\![\cdot]\!]$: jump across facets. $\rightsquigarrow$ communication between neighbors.

Solving linear systems with DG



Issues of DG methods (compared to CG)

- Breaking up continuity introduces more unknowns (`dofs`)
- Essentially all element `dofs` couple with all neighbor `dofs` $\rightsquigarrow$ even more couplings, i.e. more non-zero entries (`nzes`)
- As all element unknowns couple with neighbor, no unknowns can be eliminated, no static condensation
  [for CG: `ndofs` $\mathcal{O}(p^d)$ $\rightsquigarrow$ globally coupled `ndofs` $\mathcal{O}(p^{d-1})$]

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[1] D.A. Di Pietro, A. Ern, S. Lemaire, *An arbitrary-order and compact-stencil discretization of diffusion on general meshes [...]*. CMAM, 2014

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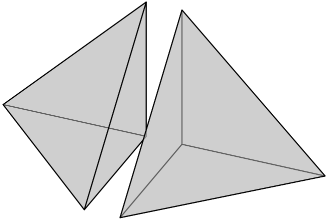

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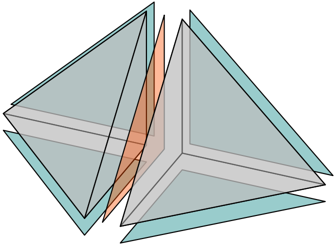
Possible Remedy: Hybrid(ized) formulations

- Hybrid DG[3]
- Hybrid High Order[1] (HHO)
- Weak Galerkin[2]



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Hybrid DG in primal formulation (Hybrid symmetric IP)

$$\mathcal{L}u = -\Delta u = f \text{ in } \Omega, \quad g = u \text{ on } \partial\Omega.$$

HDG discretization

Find $\underline{u}_h = (u_h, \lambda_h) \in V^p(\mathcal{T}_h) \times F^p_{h,D}$, s.t. $a_h(\underline{u}_h, \underline{v}_h) = \ell(\underline{v}_h) \qquad \forall \underline{v}_h = (v_h, \mu_h) \in V^p(\mathcal{T}_h) \times F^p_{h,0}$ with

$F^p_h = \{v \in L^2(\mathcal{F}_h) \mid v|_F \in \mathcal{P}^p(F) \ \forall F \in \mathcal{F}_h\}, \quad F^p_{h,D/0} = \{v \in F_h \mid v|_F = \Pi g/0 \ \forall F \in \mathcal{F}_h^{\text{bnd}}\}$, and

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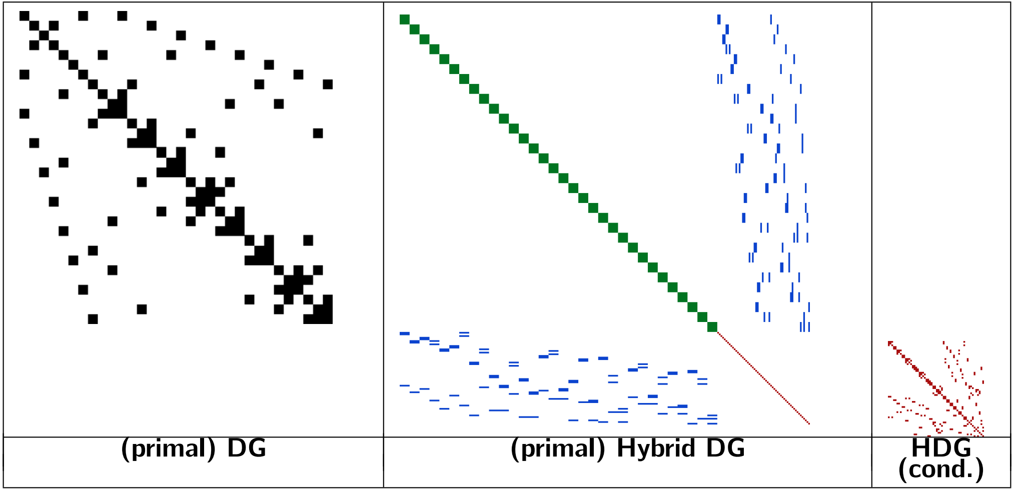
Hybrid DG: static condensation for primal DG

Solving the linear system $\hat{a}_h(\lambda_h, \mu_h) = \hat{b}_h(\mu_h)\ \forall \mu_h \in F_h$ corresponds to static condensation:



primal HDG, 2D, $p = 8$

superconvergence (gain of add. order (for diffusion dominated problems)

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Reduction of global `dofs` using ~~Hybrid~~ ⇝ Trefftz DG

Alternative to Hybridization

So far: reduce global `dofs` to facet `dofs` (static condensation)
Now: reduce global `dofs` completely (without static condensation), s.t.

- approximation (order) is preserved
- `ndofs`: $\mathcal{O}(p^d) \rightsquigarrow \mathcal{O}(p^{d-1})$

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Trefftz DG idea: Choose local PDE solutions for finite element spaces

- Replace local polynomial space by a different space (with lower dimension)
- Space contains element-local PDE solutions
- DG variational formulation to impose inter-element regularity, boundary conditions, etc...

Example: Trefftz DG for Laplace[6]

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Trefftz DG discretization

$$\mathbb{T}^p(\mathcal{T}_h) := \{v \in V^p(\mathcal{T}_h), \; \mathcal{L}v = 0 \text{ on each } K \in \mathcal{T}_h\} \subset V^p(\mathcal{T}_h).$$

Find $u_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h)$, s.t. $a_h(u_{\mathbb{T}}, v_{\mathbb{T}}) = \ell(v_{\mathbb{T}}) \qquad \forall v_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h)$.

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Error analysis (Laplace)

Céa result is inherited (coercivity)

- DG formulation consistent and $a_h(\cdot,\cdot)$ is continuous and coercive on $V^p(\mathcal{T}_h)$ w.r.t. $\|\cdot\|_{1,h}$
- both is inherited on the subspace $\mathbb{T}^p(\mathcal{T}_h) \subset V^p(\mathcal{T}_h)$
- We can apply Céa's lemma:

$$\|u - u_h\|_{1,h} \lesssim \inf_{v_h \in \mathbb{T}^p(\mathcal{T}_h)} \|u - v_h\|_{1,h} = \inf_{\substack{v_h \in V^p(\mathcal{T}_h) \\ -\Delta v_h|_{\mathcal{T}_h} = 0}} \|u - v_h\|_{1,h}$$

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Approximation

- Interpolation with avg. Taylor pol. $\mathcal{I}_K^k : H^k(K) \to \mathcal{P}^k(K)$ has $[D^\alpha \mathcal{I}^k u = \mathcal{I}^{k-|\alpha|} D^\alpha u]$

$$-\Delta u = 0 \;\Rightarrow -\Delta \mathcal{I}_K^k u = 0 \;\Rightarrow \mathcal{I}_K^k u \in \mathbb{T}^p(K)$$

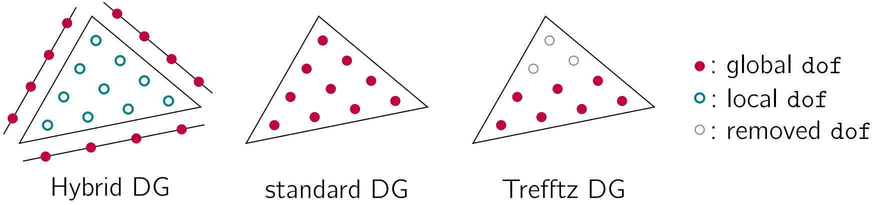
- $\mathcal{I}_K^k$ has optimal (order) approximation properties. For a solution $u \in H^m(\Omega)$ there holds

$$\Rightarrow \|u - u_h\|_{1,h} \lesssim \inf_{v_h \in \mathbb{T}^p(\mathcal{T}_h)} \|u - v_h\|_{1,h} \le \|u - \mathcal{I}_h u\|_{1,h} \lesssim h^l \, \|u\|_{H^{l+1}(\mathcal{T}_h)}, \quad l = \min\{k, m-1\}$$

Reduction of computational costs (Laplace)

Counting of `ndofs` (triangular mesh, $\mathcal{L} = -\Delta$)

- $N = \dim(V^p(\mathcal{T}_h)) = \#\mathcal{T}_h \cdot \frac{(p+1)(p+2)}{2} \sim \mathcal{O}(p^d)$,
- $L = \dim(\text{range}\,(\mathcal{L})) = V^{p-2}(\mathcal{T}_h) = \#\mathcal{T}_h \cdot \frac{(p-1)p}{2} \sim \mathcal{O}(p^d)$,
- $M = \dim(\mathbb{T}^p(\mathcal{T}_h)) = \dim(\ker\,(\mathcal{L})) = N - L = \#\mathcal{T}_h \cdot (2p+1) \sim \mathcal{O}(p^{d-1})$



Trefftz DG achieves reduction $\mathcal{O}(p^d) \rightsquigarrow \mathcal{O}(p^{d-1})$![7]

[7] We will take a look at constants later

Problems of Trefftz DG methods

Potential

- Reduction of `ndofs` without Hybridization
- Interesting for instance for unfitted FEM

Disadvantages

- Need to implement a new basis for each diff operator $\mathcal{L}$ / PDE
- Conditioning of new basis often problematic

Limitations

- Not directly suitable for inhomogeneous equations $f \neq 0$
- Not directly suitable for non-constant coefficients, e.g. $\mathcal{L} = \mathrm{div}(\alpha \nabla \cdot)$

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Overall: Method not flexible, used only in special cases

$\rightsquigarrow$ Can we turn Trefftz into a (more) general purpose tool?

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Scalar, linear case with (suitable) polynomial Trefftz spaces

Assumptions

- scalar PDE
- $f = 0$
- no "competing" derivatives in the same direction: $\mathcal{L} = \sum_{l=1}^{d} \alpha_l \partial_{x_l}^{\beta_l}$ for $\alpha_l \in \mathbb{R}$ and $\beta_l \in \mathbb{N}$
- constant coefficients and straight elements (or no reference element mapping)

Examples (assumption fulfilled):

- $\mathcal{L} = -\Delta$
- $\mathcal{L} = b \cdot \nabla$ for $b \in \mathbb{R}^d$
- $\mathcal{L} = \partial_t + b \cdot \nabla$
- $\mathcal{L} = \partial_t - \Delta$

Examples (assumption not fulfilled):

- $\mathcal{L} = -\Delta \pm \mathrm{id}$
- $\mathcal{L} = -\Delta + b \cdot \nabla$
- $\mathcal{L} = -\operatorname{div}(\alpha \nabla \cdot)$, α not constant

Construction of a Trefftz Embedding I/II

Galerkin isomorphisms

- standard DG: $\mathcal{G} : \mathbb{R}^N \to V^p(\mathcal{T}_h)$, $\quad \mathbf{x} \to \sum_{i=1}^N \mathbf{x}_i \phi_i$, with $\{\phi_i\}$ basis of $V^p(\mathcal{T}_h)$
- Trefftz DG: $\mathcal{G}_\mathbb{T} : \mathbb{R}^M \to \mathbb{T}^p(\mathcal{T}_h)$, $\quad \mathbf{x} \to \sum_{j=1}^M \mathbf{x}_j \psi_j$, with $\{\psi_j\}$ basis of $\mathbb{T}^p(\mathcal{T}_h)$

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We want to avoid setting up the basis $\{\psi_j\}$ from scratch! ($\rightsquigarrow$ flexibility)

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Galerkin isomorphisms

- standard DG: $\mathcal{G} : \mathbb{R}^N \to V^p(\mathcal{T}_h)$, $\quad \mathbf{x} \to \sum_{i=1}^N \mathbf{x}_i \phi_i$, with $\{\phi_i\}$ basis of $V^p(\mathcal{T}_h)$
- Trefftz DG: $\mathcal{G}_{\mathbb{T}} : \mathbb{R}^M \to \mathbb{T}^p(\mathcal{T}_h)$, $\quad \mathbf{x} \to \sum_{j=1}^M \mathbf{x}_j \psi_j$, with $\{\psi_j\}$ basis of $\mathbb{T}^p(\mathcal{T}_h)$

We want to avoid setting up the basis $\{\psi_j\}$ from scratch! ($\rightsquigarrow$ flexibility)

Trefftz DG basis can be represented through DG basis

For any basis $\{\psi_j\}$ we have $\psi_j \in V^p(\mathcal{T}_h) \;\Rightarrow\; \psi_j = \sum_{j=1}^N \mathbf{T}_{ij} \phi_i,\ j = 1, .., M$, for $\mathbf{T} \in \mathbb{R}^{N \times M}$.

Task: Compute $\mathbf{T}$ (so that $\mathcal{G}_{\mathbb{T}}(\mathbf{x}) = \mathcal{G}(\mathbf{T}\mathbf{x})$)!

Construction of a Trefftz Embedding II/II

$$V^p(\mathcal{T}_h) \supset \mathbb{T}^p(\mathcal{T}_h) = \ker(\mathcal{L}) \quad \Rightarrow \quad \mathbb{R}^N \supset \ker(\mathbf{W}) = \mathbf{T} \cdot \mathbb{R}^M$$

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$\mathbf{W} \in \mathbb{R}^{N \times N}$ characterizes $\ker(\mathcal{L})$ as $\ker(\mathcal{L}) = \mathcal{G}(\ker(\mathbf{W}))$ with (for now):

$$(\mathbf{W})_{ij} = w_h(\phi_j, \phi_i) = \langle \mathcal{L}\phi_j, \mathcal{L}\phi_i \rangle_{0,h} = \sum_{K \in \mathcal{T}_h} \langle \mathcal{L}\phi_j, \mathcal{L}\phi_i \rangle_K, \quad i, j = 1, \ldots, N$$

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Compute $\mathbf{W}$ (block-diag) and compute $\ker(\mathbf{W})$ numerically, e.g. by SVD (alternative: QR)

- $$\mathbf{W} = \begin{pmatrix} | & & | & | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_L & \mathbf{u}_{L+1} & \dots & \mathbf{u}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_L & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{v}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_L^T & \text{---} \\ \text{---} & \mathbf{v}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_N^T & \text{---} \end{pmatrix}$$

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  (costs: $\mathcal{O}(\#\mathcal{T}_h \cdot M_K^3) = \mathcal{O}(\#\mathcal{T}_h \cdot p^{3d})$)
- Columns of $\mathbf{T}$ are orthogonal $\Rightarrow \mathbf{T}^T\mathbf{T} = \mathbf{I}_{M \times M}$

Setup of Embedded Trefftz DG linear systems

Standard DG setting: matrix/vector

$$(\mathbf{A})_{ij} = a_h(\mathcal{G}(\mathbf{e}_j), \mathcal{G}(\mathbf{e}_i)) = a_h(\phi_j, \phi_i) \; i, j = 1, \ldots, N,$$
$$(\boldsymbol{\ell})_i = \ell(\mathcal{G}(\mathbf{e}_i)) = \ell(\phi_i) \; i = 1, \ldots, N$$

Setup of Trefftz DG linear system (exploiting emb. matrix $\mathbf{T}$)

1. Assemble $\mathbf{A}$, $\boldsymbol{\ell}$ (standard DG)
2. Setup $\mathbf{T}$ (Trefftz embedding matrix)
3. Setup $\tilde{\mathbf{A}} = \mathbf{T}^T\mathbf{A}\mathbf{T}$, $\tilde{\boldsymbol{\ell}} = \mathbf{T}^T\boldsymbol{\ell}$
4. Solve: Find $\mathbf{u}_{\mathbb{T}} (= \mathcal{G}_{\mathbb{T}}^{-1}(u_{\mathbb{T}}))$ so that

$$\mathbf{T}^T\mathbf{A}\mathbf{T}\,\mathbf{u}_{\mathbb{T}} = \mathbf{T}^T\boldsymbol{\ell}.$$

Benefits of Embedded Trefftz DG

So far:

Embedded Trefftz DG ...

1. ... facilitates implementation of existing polynomial Trefftz methods
2. ... is an implementation trick (same solution as "direct" Trefftz method)

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Next:

Claim: Embedded Trefftz DG ...

1. ... inherites conditioning properties from DG scheme
2. ... allows to treat inhomogeneous PDEs
3. ... allows to conveniently implement weak Trefftz spaces
   $\rightsquigarrow$ treat PDEs where no (suitable) polynomial Trefftz spaces exists

Conditioning

The resulting linear system is well controlled in terms of its conditioning.

Lemma (Conditioning of the embedded Trefftz method)

$$\kappa_2(\tilde{\mathbf{A}}) = \kappa_2(\mathbf{T}^T\mathbf{A}\mathbf{T}) \leq \kappa_2(\mathbf{A}).$$

Proof.

By construction of $\mathbf{T}$ all its column vectors are orthogonal. □

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Proof.

By construction of **T** all its column vectors are orthogonal. $\square$

$\rightsquigarrow$ we can build on well-developed DG spaces (an implementations)

Inhomogeneous PDEs I/II

$$\mathcal{L}u = f \text{ in } \Omega, \quad f \in L^2(\Omega), \qquad + \quad \text{bound. cond.}$$

Standard DG

Find $u_h \in V^p(\mathcal{T}_h)$, s.t. $a_h(u_h, v_h) = \ell(v_h) = g_{\text{b.c.}}(v_h) + \langle f, v_h \rangle_{0,h} \qquad \forall v_h \in V^p(\mathcal{T}_h)$.

Trefftz DG

- Ansatz space solves (element-wise) homogeneous equation $\mathcal{L}v = 0$
- Replacing $V^p(\mathcal{T}_h)$ with $\mathbb{T}^p(\mathcal{T}_h)$ will not work
- Homogenization requires a particular solution

Inhomogeneous PDEs II/II

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Embedded Trefftz DG

- Compute discrete particular solution $u_{h,f} \in V^p(\mathcal{T}_h)$:

$$\mathcal{L}u_{h,f} \approx f \quad \rightsquigarrow \quad w_h(u_{h,f}, v) = (f, \mathcal{L}v)_{0,h} \;\; \forall v \in V^p(\mathcal{T}_h) \;\Rightarrow\; \mathbf{u}_f = \mathbf{W}^\dagger \mathbf{f}$$

- $\mathbf{W}^\dagger$ available from SVD (or QR) (element-wise, in parallel)

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- $\mathbf{W}^\dagger$ available from SVD (or QR) (element-wise, in parallel)
- After computing $u_{h,f}$ solve: Find $u_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h)$ so that

$$a_h(u_{\mathbb{T}}, v_{\mathbb{T}}) = \ell(v_{\mathbb{T}}) - a_h(u_{h,f}, v_{\mathbb{T}}) \;\; \forall\, v_{\mathbb{T}} \in \mathbb{T}^p(\mathcal{T}_h). \tag{1}$$

This translates to the solution of the linear system

$$\mathbf{T}^T \mathbf{A} \mathbf{T} \mathbf{u}_{\mathbb{T}} = \mathbf{T}^T(\boldsymbol{\ell} - \mathbf{A}\mathbf{u}_f). \tag{2}$$

Overview

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more details: C.L. & P. Stocker, *Embedded Trefftz DG methods*, https://arxiv.org/abs/2201.07041

Non-polynomial Trefftz spaces

Many problems don't have suitable polynomial Trefftz spaces

Examples:

- $\mathcal{L} = -\Delta \pm \mathrm{id}, \quad p \in \mathcal{P}^p, \ \mathcal{L}p = 0 \quad \Rightarrow p = \mp \Delta p \in \mathcal{P}^{p-2} \quad \Rightarrow \ p = 0$ ↯
- $\mathcal{L} = -\Delta + b \cdot \nabla$
- $\mathcal{L} = -\operatorname{div}(\alpha \nabla \cdot)$, α not constant

[8] C. J. Gittelson, R. Hiptmair, and I. Perugia, *Plane wave discontinuous Galerkin methods: Analysis of the h-version*, ESAIM:M2AN, 2009

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How to do Trefftz in these cases?

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Trefftz DG based on plane waves[8]

For Helmholtz $(-\Delta - \omega^2\,\mathrm{id})$ Plane Wave DG (a Trefftz DG) spaces exist:

$$\mathbb{T}^p = \{e^{-i\omega(d_j \cdot x)} \text{ s.t. } j = 0, \ldots, k\}$$

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Quasi-Trefftz Methods[9]

Let $\mathcal{L}_\alpha$ be diff. operator depending on a (element-wise) smooth α, define quasi-Trefftz space

$$\mathbb{QT}^p := \{v \in V^p(\mathcal{T}_h) \mid T^{p-q}_{(x_{\text{center}})}(\mathcal{L}_\alpha v) = 0\}$$

---

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Weak Trefftz spaces

Observation

In Embedded Trefftz DG methods the Trefftz condition $\mathcal{L}v = 0$ has been realized through

$$w_h(v, w) = \langle \mathcal{L}v, \mathcal{L}w \rangle_{0,h} = 0 \quad \forall w \in V^p(\mathcal{T}_h) \iff \|\mathcal{L}v\|_{0,h} = 0$$

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Weak Trefftz condition

Now, we relax the condition by changing the test space

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with Π_W the L^2 projection into $W(\mathcal{T}_h)$ ($W(\mathcal{T}_h) := \mathcal{L}V^p(\mathcal{T}_h)$ recovers "strong" Trefftz).

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$\implies$ Weak Trefftz space: $\quad \mathbb{T}^p(\mathcal{T}_h) = \{v \in V^p(\mathcal{T}_h) \mid \Pi_W \mathcal{L}v = 0\}$

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Numerical analysis:

- stability: clear in coercive case, open in the general case (case by case),
- approximation: open (unless equiv. to other Trefftz DG methods)

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Algorithmic structure

Pseudo-Code

Require: Basis functions $\{\phi_i\}_i$, DG formulation (a_h, l), operator $\mathcal{L}$, space W, trunc. parameter ε, r.h.s. f

function dg matrix
  $(\mathbf{A})_{ij} = a_h(\phi_j, \phi_i)$
  $(\boldsymbol{\ell})_i = \ell(\phi_i)$
for $K \in \mathcal{T}_h$ **do**
  $(\mathbf{W}_K)_{ij} = \langle \mathcal{L}\phi_j, \varphi_i \rangle_{0,h}$
  $\mathbf{T}_K = \ker_h(\varepsilon; \mathbf{W}_K)$
  if $f \neq 0$ **then**
    $(\mathbf{w}_K)_i = \langle f, \varphi_i \rangle_{0,h}$
    $(\mathbf{u}_f)_K = \mathbf{W}_K^{\dagger} \mathbf{w}_K$
Solve $\mathbf{T}^T \mathbf{A} \mathbf{T}\, \mathbf{u}_{\mathbb{T}} = \mathbf{T}^T(\boldsymbol{\ell} - \mathbf{A}\mathbf{u}_f)$
$\mathbf{u}_h = \mathbf{T}\mathbf{u}_{\mathbb{T}} + \mathbf{u}_f$
output $\mathbf{u}_h$

NGSolve

```
def Solve(mesh, order, dgscheme,
          L, W, eps, rhs):
  V = L2(mesh,order=order,dgjumps=True)
  uh = GridFunction(V)
  a,f = dgscheme(V)
  u,v = V.TnT()
  wh = L(u)*w*dx
  rhsw = rhs*w*dx
  T, uf = TrefftzEmbedding(wh,V,rhsw,eps,W)
  Tt = T.CreateTranspose()
  TA = Tt@a.mat@T
  ut = TA.Inverse()*(Tt*(f.vec-a.mat*uf))
  uh.vec.data = T*ut + uf
  return uh
```

Algorithmic complexity: A rough comparison

- direct solver
- $N_{el} := \#\mathcal{T}_h \sim h^{-d}$
- p-scaling (no constants)

Costs:	Standard DG	Trefftz DG	Embedded Trefftz DG	Hybrid DG
Vector representation:				
total `ndofs` stored	$\sim N_{el}\, p^d$	$\sim N_{el}\, p^{d-1}$	$\sim N_{el}\, p^d$	$\sim N_{el}\, p^d$
globally coupled `ndofs`	$\sim N_{el}\, p^d$	$\sim N_{el}\, p^{d-1}$	$\sim N_{el}\, p^{d-1}$	$\sim N_{el}\, p^{d-1}$
Setup linear systems:				
`nzes` $\mathbf{A}$	$\sim N_{el}\, p^{2d}$	$\sim N_{el}\, p^{2d-2}$	$\sim N_{el}\, p^{2d}$	$\sim N_{el}\, p^{2d}$
Additional costs:			Setup $\mathbf{T}$:	static cond.:
	—	—	$\sim N_{el}\, p^{3d}$	$\sim N_{el}\, p^{3d}$
Solving linear systems:				
global matrix	$\mathbf{A}$	$\mathbf{A}$	$\mathbf{T}^T\mathbf{A}\mathbf{T}$	$\mathbf{S}$
`nzes`	$\sim N_{el}\, p^{2d}$	$\sim N_{el}\, p^{2d-2}$	$\sim N_{el}\, p^{2d-2}$	$\sim N_{el}\, p^{2d-2}$

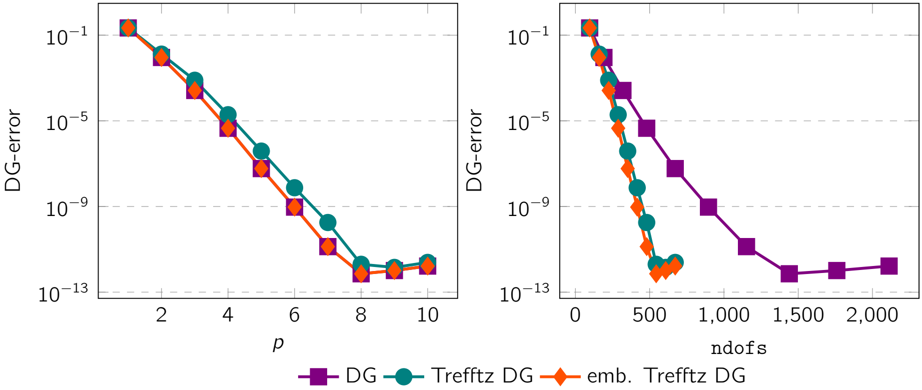
Overview

more details: C.L. & P. Stocker, *Embedded Trefftz DG methods*, `https://arxiv.org/abs/2201.07041`

Laplace: $-\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$

- Symmetric IP formulation
- Nested simplicial meshes
- Manufactured solution:

$$2D: \quad u = \exp(x)\sin(y), \Omega = (0,1)^2.$$



[6] R. Hiptmaier, A. Moiola, I. Perugia, C. Schwab, *Approximation by harmonic polynomials [..] and exponential convergence of Trefftz hp-dGFEM*, ESAIM, 2014

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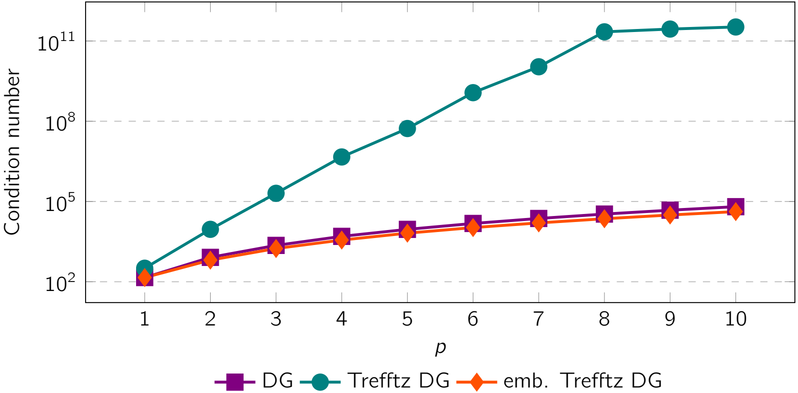

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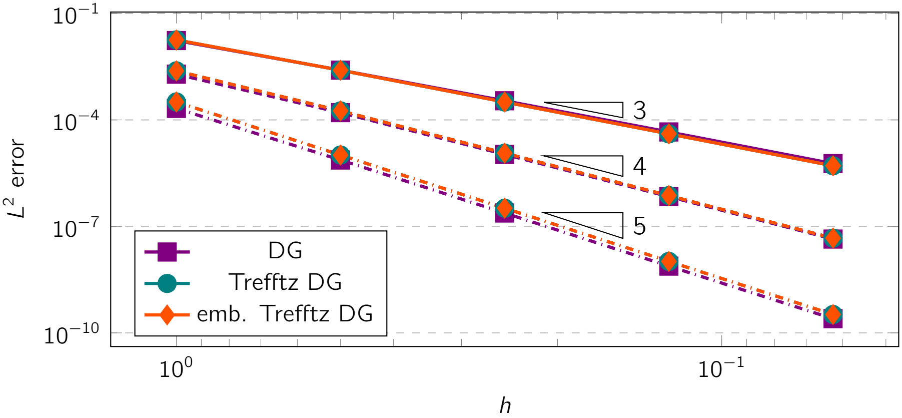
$$3D: \quad u = \exp(x+y)\sin(\sqrt{2}z), \Omega = (0,1)^3, \quad p = 2,3,4$$



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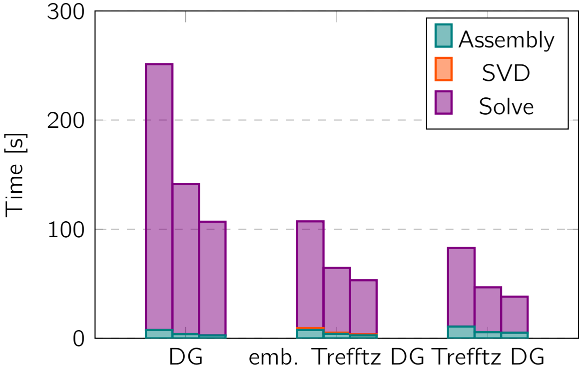

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More numerical results for problems with polynomial Trefftz DG spaces

- Poisson $\rightsquigarrow$[10]
- Space-Time wave equation $\rightsquigarrow$[10]

Both show similar perfomance.

$\rightsquigarrow$ let's take a look at non-polynomial Trefftz DG spaces

[6] R. Hiptmaier, A. Moiola, I. Perugia, C. Schwab, *Approximation by harmonic polynomials [..] and exponential convergence of Trefftz hp-dGFEM*, ESAIM, 2014

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Helmholtz: $\mathcal{L}u = -\Delta u - \omega^2 u = 0$ in $\Omega = (0,1)^2$, $\partial_n u + iu = g$ on $\partial\Omega$

Setup: Schemes and solution

- DG scheme from the literature[11] with spaces:
  1. polynomial DG space $V^p(\mathcal{T}_h)$
  2. Plane Wave DG (PWDG) space (non-polynomial Trefftz DG space):

$$\mathbb{T}^k = \{e^{-i\omega(d_j\cdot x)} \text{ s.t. } j = 0, \dots, 2k\}, \qquad \dim(\mathbb{T}^p(\mathcal{T}_h)) = \#\mathcal{T}_h \cdot (2p+1)$$

     with d_j: evenly spaced direction vectors, $d_j = (\cos(\pi\frac{j}{2p+1}), \sin(\pi\frac{j}{2p+1}))^T$, $j = 0, .., 2k$
  3. (Embedded) Weak Trefftz space:

$$\mathbb{WT}^p = \{v_h \in V^p(\mathcal{T}_h) \mid \Pi_W \mathcal{L} v_h = 0\}$$

     with $W = V^{p-2}(\mathcal{T}_h)$, s.t. $\langle \mathcal{L}v_h, w_h\rangle_{0,h} = 0 \;\forall\, w_h \in W$, $\dim(\mathbb{WT}^p(\mathcal{T}_h)) = \#\mathcal{T}_h \cdot (2p+1)$
- Manufactured solution :

$$u = H_0^{(1)}(\omega|\mathbf{x} - \mathbf{x}_0|), \quad \mathbf{x}_0 = (-0.25, 0), \quad H_0^{(1)} \text{zero-th order Hankel function of first kind}$$

[11] O. Cessenat, B. Després, *Application of an ultra weak variational formulation of elliptic pdes to the two-dimensional Helmholtz problem*, SINUM, 1998.

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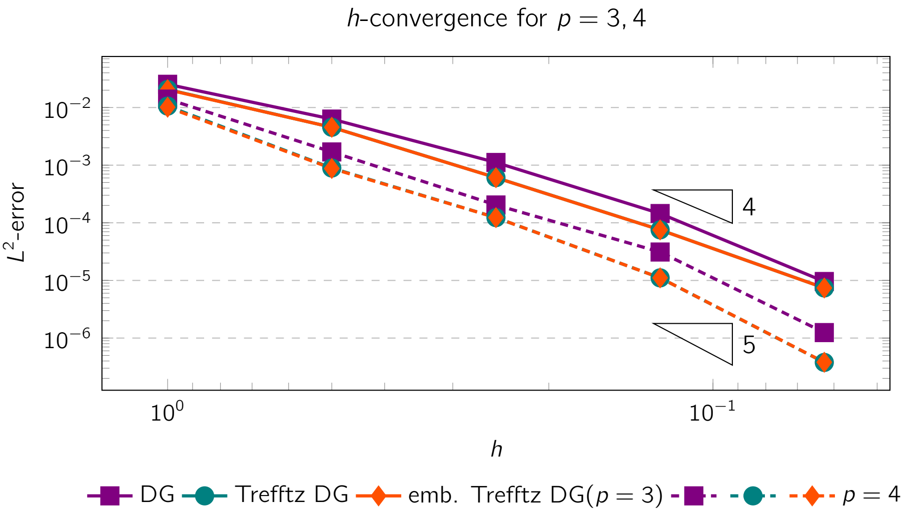

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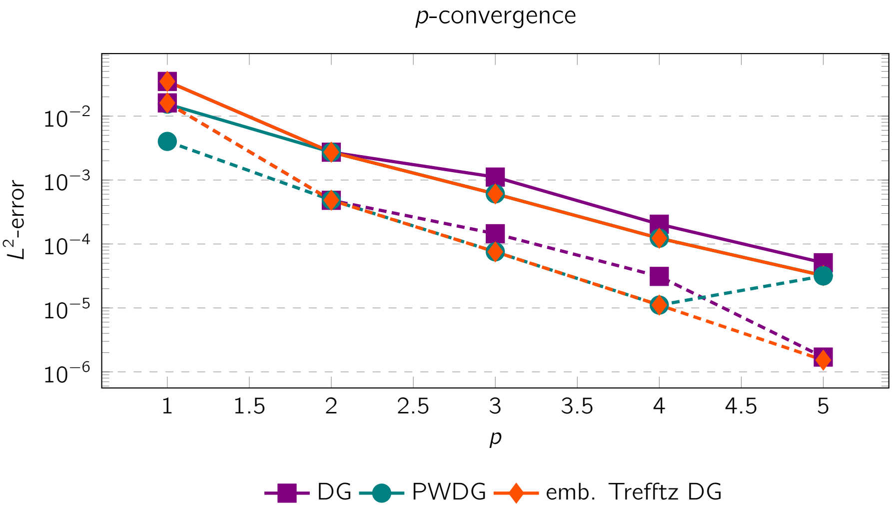

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Linear hyperbolic transport: $\mathbf{b} \cdot \nabla u = f$ in Ω, $u = u_D$ on $\partial\Omega_{\text{in}}$

Setup (manufactured solution, 3D):

$$\Omega = (0,1)^3, \ \mathbf{b} = (-\sin(x_2), \cos(x_1), x_1)^T, \partial\Omega_{\text{in}} := \{\mathbf{x} \in \partial\Omega \mid \mathbf{b} \cdot n_x < 0\}, u = \sin(x_1)\sin(x_2)\sin(x_3)$$

Standard DG Upwind discretization ($\hat{u}(\mathbf{x}) = \lim_{h \to 0^+} u(\mathbf{x} - \mathbf{b}h)$),

$$a_h(u,v) = \sum_{K \in \mathcal{T}_h} \left\{ \int_K -u \, \mathbf{b} \cdot \nabla v \, dx + \int_{\partial K \setminus \partial\Omega\text{in}} \mathbf{b}_n \hat{u} v \, ds \right\}, \quad \ell(v) = \sum_{K \in \mathcal{T}_h} \int_K fv \, dx - \int_{\partial\Omega\text{in}} \mathbf{b}_n u_D v \, ds$$

Spaces:

1. standard DG space $V^p(\mathcal{T}_h)$
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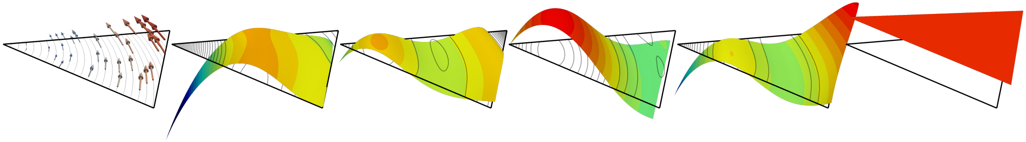
Illustration of Weak Trefftz basis (2D, $k = 4$, $\dim(\mathbb{WT}^p(K)) = k + 1$):



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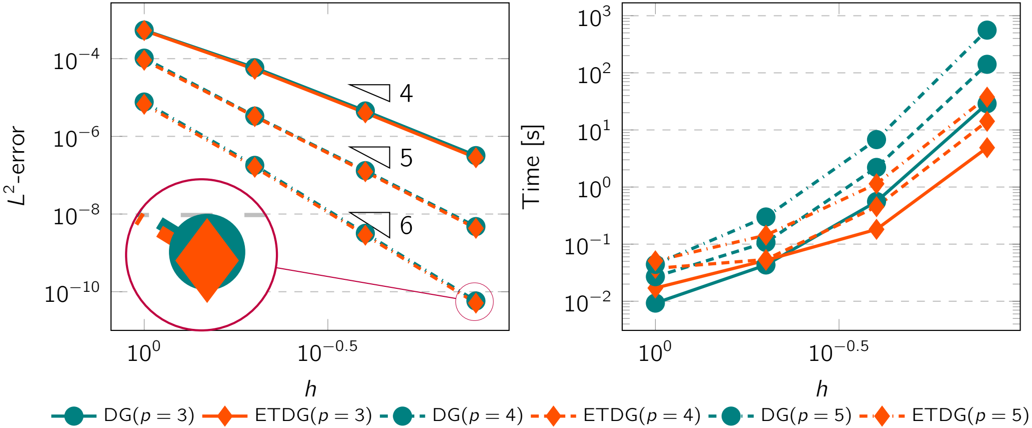

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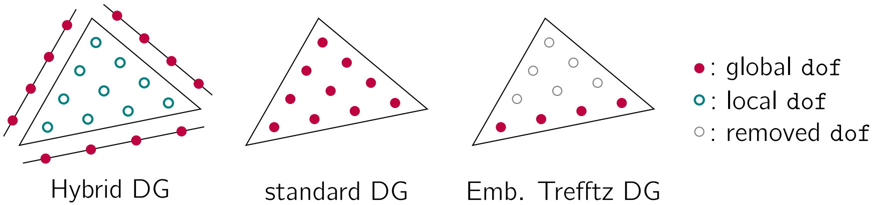
Explanation for strong performance gains (sketch)



Overview

Repetition of Discontinuous Galerkin (DG) und Hybrid DG methods

- Discontinuous Galerkin (DG)
- Hybrid DG

Trefftz DG and Embedded Trefftz DG methods

- Trefftz DG
- Embedded Trefftz DG methods for polynomial Trefftz methods
- Embedded Trefftz DG methods beyond polynomial Trefftz methods

Algorithmic aspects & Numerical examples

- Laplace, Poisson, Helmholtz, linear advection
- Comparison to DG/HDG

Conclusion & Outlook

more details: C.L. & P. Stocker, *Embedded Trefftz DG methods*, `https://arxiv.org/abs/2201.07041`

Some number crunching

d	p	ndofs DG	HDG	TDG(1)	TDG(2)	nzes DG	HDG	TDG(1)	TDG(2)
2	0	54	91	54	54	196	415	196	196
2	1	162	182	108	162	1,764	1,660	784	1,764
2	2	324	273	162	270	7,056	3,735	1,764	4,900
2	3	540	364	216	378	19,600	6,640	3,136	9,604
2	4	810	455	270	486	44,100	10,375	4,900	15,876
2	5	1,134	546	324	594	86,436	14,940	7,056	23,716
3	0	729	1,612	729	729	3,337	10,360	3,337	3,337
3	1	2,916	4,836	2,187	2,916	53,392	93,240	30,033	53,392
3	2	7,290	9,672	4,374	6,561	333,700	372,960	120,132	270,297
3	3	14,580	16,120	7,290	11,664	1,334,800	1,036,000	333,700	854,272
3	4	25,515	24,180	10,935	18,225	4,087,825	2,331,000	750,825	2,085,625
3	5	40,824	33,852	15,309	26,244	10,464,832	4,568,760	1,471,617	4,324,752

ndofs: globally coupled ndofs

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`ndofs`: globally coupled `ndofs`

Observations (`dofs` per entity (facet/el.) / coupled blocks)

- Trefftz DG always beats DG
- Trefftz DG shows no "low order overhead" as Hybrid DG
- Trefftz DG also beats Hybrid DG (unless superconvergence tweaks are possible!)
- For first order problems: Trefftz DG beats Hybrid DG by factor ≈ 2 (in `ndofs`)

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Benefits of Embedded Trefftz DG

- Improve existing polynomials Trefftz DG methods:
  - simplify implementation: no explicit basis needed
  - reasonable conditioning guaranteed by DG space
  - handle inhomogeneous r.h.s. naturally
- Allow for new polynomial Trefftz DG methods:
  - Generic way to construct Weak Trefftz DG methods with $\Pi_W \mathcal{L} v = 0$

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Outlook: Potential for different problems/settings

- Heat equation (space-time DG), Vector PDEs (Stokes, Elasticity, Galbrun's equation),
- Non-linear problems / time stepping (generic Trefftz DG within each iteration),
- Unfitted DG (bad cuts don't combine well with HDG ϟ),
- $\mathcal{L}$ in $\mathbf{ker}(\mathcal{L})$ does not (only) need to be the diff op. of the volume PDE,
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`NGSTrefftz`: `http://github.com/PaulSt/ngstrefftz` (interactive demos)

arXiv preprint: `https://arxiv.org/abs/2201.07041`

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Thank you for your attention!

HDG and superconvergence (the gain of two orders)

Costs of HDG

- Dominating costs of HDG depend on the `ndofs` on the skeleton.
- In Hybrid Mixed methods we can obtain a post-processed solution in $V^{p+1}(\mathcal{T}_h)$ (with $\|\cdot\|_{L^2(\Omega)}$-error $\lesssim h^{p+2}$) from global linear systems with F_h^p
- Similar things[12] can be done by reducing the facet degree: $F_h^p \rightsquigarrow F_h^{p-1}$

HDG discretization with projected jumps

Find $\underline{u}_h = (u_h, \lambda_h) \in V^p(\mathcal{T}_h) \times F_{h,D}^{p-1}$, s.t. $a_h(\underline{u}_h, \underline{v}_h) = \ell(\underline{v}_h) \qquad \forall \underline{v}_h = (v_h, \mu_h) \in V^p(\mathcal{T}_h) \times F_{h,0}^{p-1}$ with

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[12] for Poisson or in the diffusion dominated case

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Π: L^2 projection on F_h^{p-1}.

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$F_h^{p-1} = \{v \in L^2(\mathcal{F}_h) \mid v|_F \in \mathcal{P}^{p-1}(F)\ \forall F \in \mathcal{F}_h\}$, $\quad F_{h,D/0} = \{v \in F_h \mid v|_F = \Pi g/0\ \forall F \in \mathcal{F}_h^{\text{bnd}}\}$, and

$$a_h^\star(\underline{u}_h, \underline{v}_h) = \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h \nabla v_h \, dx + \int_{\partial K} \underbrace{-\partial_n u_h \Pi [\![\underline{v}_h]\!]}_{\text{consistency}} \underbrace{-\partial_n v_h \Pi [\![\underline{u}_h]\!]}_{\text{symmetry}} \underbrace{+ \alpha p^2 h^{-1} \Pi [\![\underline{u}_h]\!] \Pi [\![\underline{v}_h]\!]}_{\text{stability}} \, ds$$

Π: L^2 projection on F_h^{p-1}.

---

[12] for Poisson or in the diffusion dominated case

HDG and superconvergence (the gain of two orders)

Costs of HDG

- Dominating costs of HDG depend on the `ndofs` on the skeleton.
- In Hybrid Mixed methods we can obtain a post-processed solution in $V^{p+1}(\mathcal{T}_h)$ (with $\|\cdot\|_{L^2(\Omega)}$-error $\lesssim h^{p+2}$) from global linear systems with F_h^p
- Similar things[12] can be done by reducing the facet degree: $F_h^p \rightsquigarrow F_h^{p-1}$

HDG discretization with projected jumps

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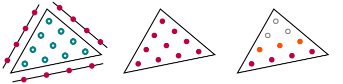
Π: L^2 projection on F_h^{p-1}. All boundary terms only appear as order $p-1$ polynomials.

[12] for Poisson or in the diffusion dominated case

What are the global couplings?

(assuming full polynomials, scalar case, simplicial mesh)



$\bullet$: $\begin{cases} \text{removed } \texttt{dof} & \text{if } q^{\text{diff}} = 1, \\ \text{global } \texttt{dof} & \text{else.} \end{cases}$

- For weak Trefftz: Reduction of `ndofs` depends on $\mathcal{L}$
  - 2^{nd} order operator: $\dim(\mathcal{P}^p(\mathbb{R}^d)) = \binom{p+d}{p} \rightsquigarrow \overbrace{\binom{p+d-1}{p}}^{\dim(\mathcal{P}^p(\mathbb{R}^{d-1}))} + \overbrace{\binom{p+d-2}{p-1}}^{\dim(\mathcal{P}^{p-1}(\mathbb{R}^{d-1}))}$

    2D: $(p+1)(p+2)/2 \rightsquigarrow 2p+1$, 3D: $(p+1)(p+2)(p+3)/6 \rightsquigarrow (p+1)^2$

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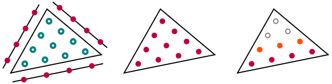

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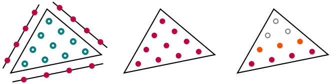

Hybrid DG standard DG Emb. Trefftz DG

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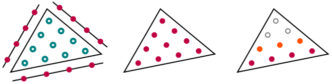

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- both have $n = \mathcal{O}(p^{d-1})$ unknowns per element, respectively facet:
  - (simplex case) an element has d neighbors $\rightsquigarrow$ $d+1$ blocks involved in coupling
  - (simplex case) a facet has $2d$ neighbor facets $\rightsquigarrow$ $2d+1$ blocks involved in coupling

A priori error analysis of Embedded Trefftz DG

Stability / Céa (underlying DG formulation is consistent, continuous)

- Coercive case (inherited):

$$\|u - u_h\|_h \lesssim \inf_{\substack{v_h \in V^p(\mathcal{T}_h) \\ \Pi_W \mathcal{L} v_f = \Pi_W f}} \|u - v_h\|_h$$

- Non-coercive case: case dependent, but norm control $\|\Pi_W \mathcal{L} v\|_{0,h}$ for free

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Approximation

- depends on W [balances approximation vs. efficiency]
- for existing Trefftz spaces optimal results are known (e.g. based on avg. Taylor pol.)
  (e.g. acoustic wave equation in time-domain, Laplace/Poisson)
- general case: open