Space-time discretizations with unfitted FEM

for moving domain problems

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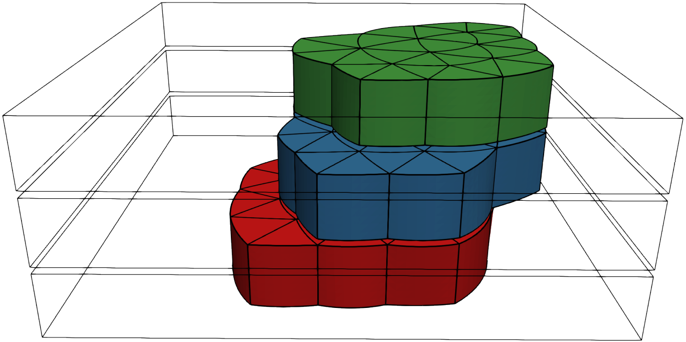
November 8, 2022, Work group seminar M. Bause, BWU Hamburg



Content

Section 1

Motivation

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Motivation

Subsection 1.1

Unfitted FEM

Motivation: Two phase flow (e.g. oil droplet in water)

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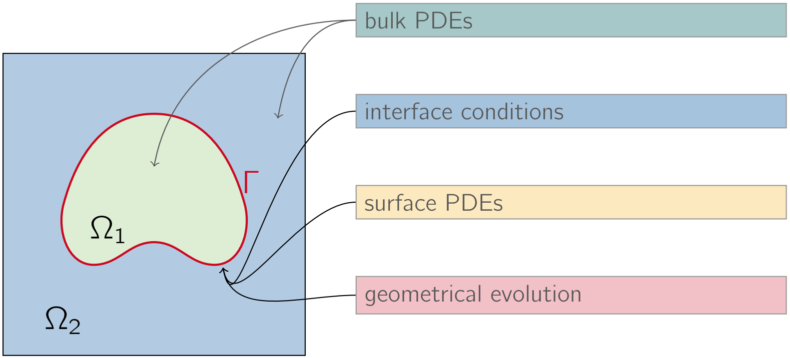
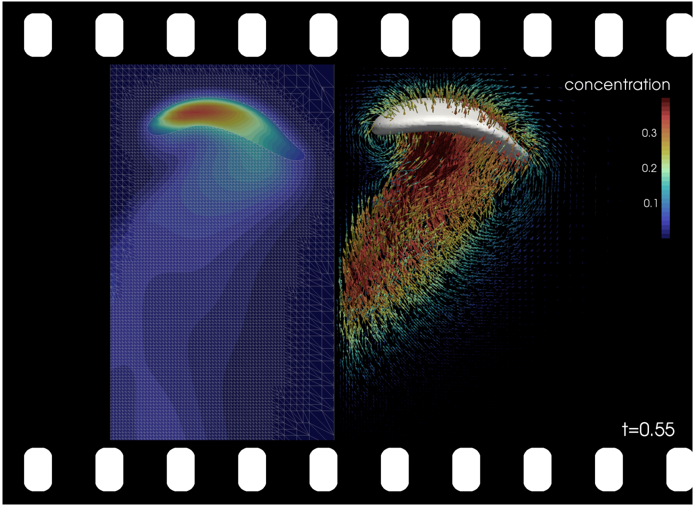


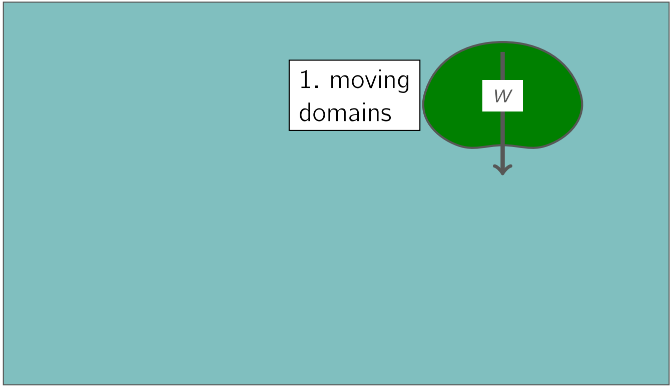
- evolution of **complex geometry**
- sub-problems are coupled (**nonlinear**)

Example configuration (movie)



[1] C.L.. The Nitsche XFEM-DG space-time method and its implementation in three space dimensions. SISC, 2015.

Motivation: body-fitted vs. unfitted



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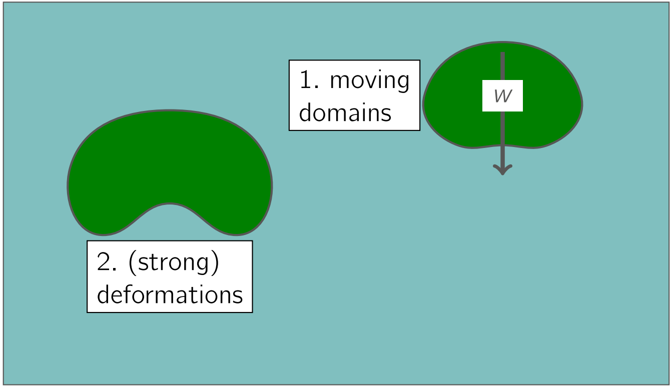

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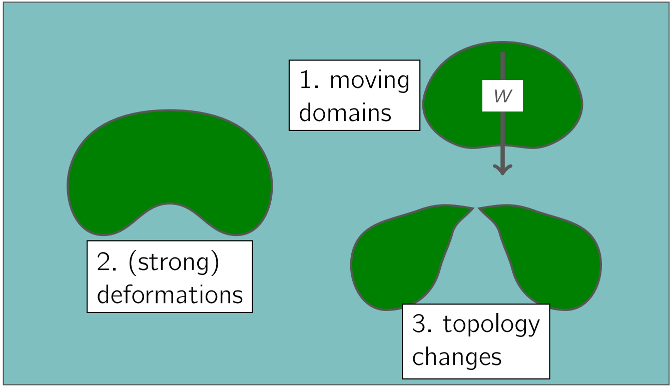

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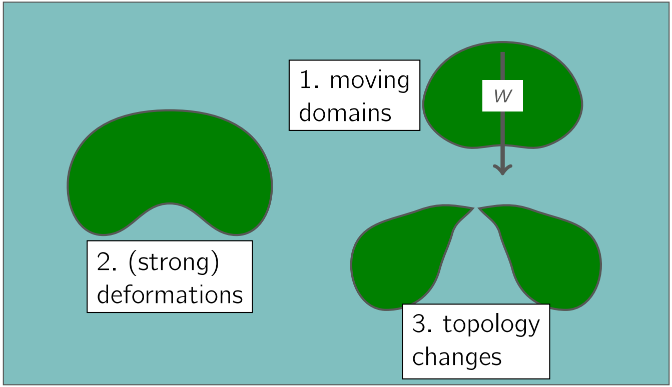

With fitted meshes?

Motivation: body-fitted vs. unfitted

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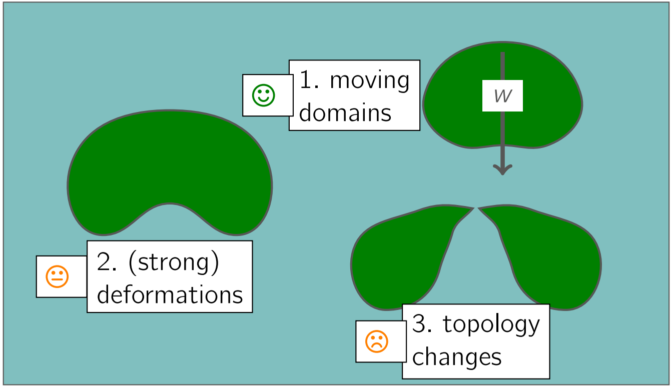
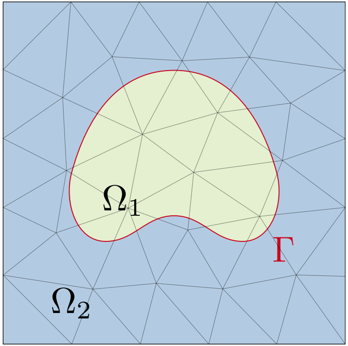
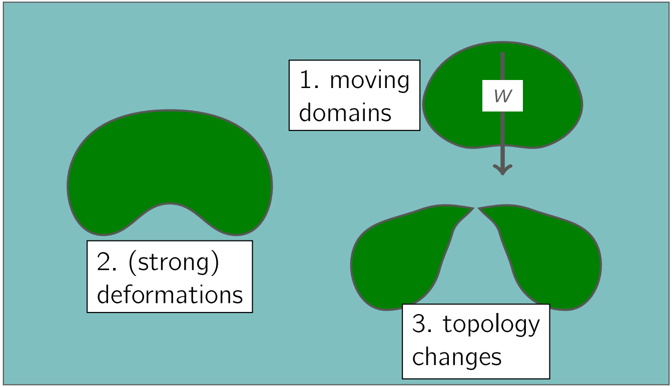


With fitted meshes? $\rightsquigarrow$ possible, but cumbersome

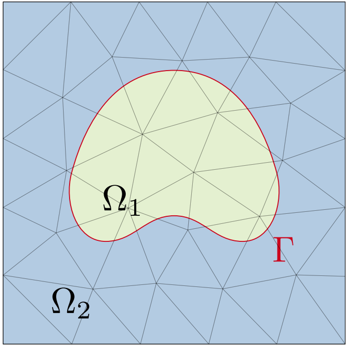
Motivation: body-fitted vs. unfitted

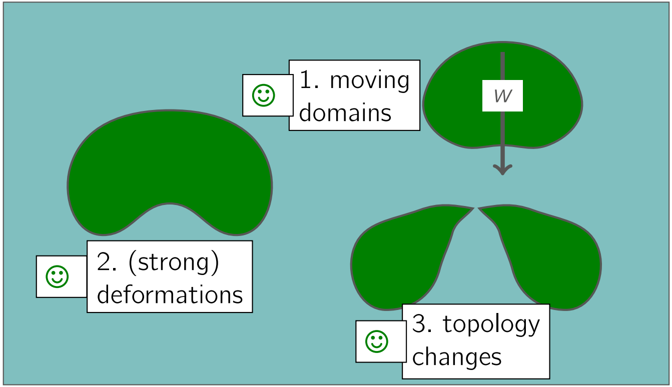




Idea of (geometrically) unfitted discretizations:
remove burden of fitted meshes (generation/tracking/remeshing)
by decoupling mesh and geometry (e.g. level set)

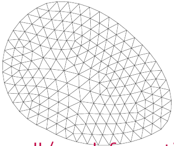
Motivation: body-fitted vs. unfitted



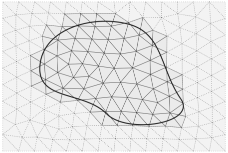


Idea of (geometrically) unfitted discretizations:
remove burden of fitted meshes (generation/tracking/remeshing)
by decoupling mesh and geometry (e.g. level set) $\rightsquigarrow$ flexible geometry handling,
New challenges: shape irregular cuts, numerical integration, time integration

Rough Comparison: fitted and unfitted discretizations



- results for small/no deformation:
  - high order accuracy
  - efficient linear solver concepts
  - robust implementation
  - rigorous error analysis, ...



! important key properties of standard FEM have to be re-established:

- stable (time) discretization
- impl. of boundary conditions
- robust numerical integration
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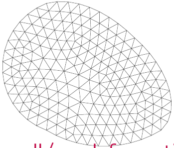
Rough Comparison: fitted and unfitted discretizations



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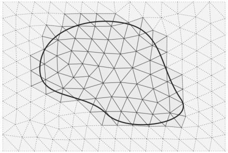

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Rough Comparison: fitted and unfitted discretizations

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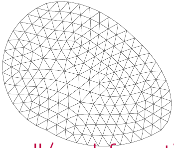

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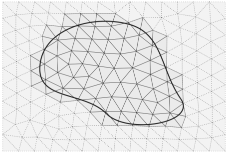
- Often mesh generation for complex geometries is very expensive



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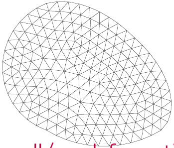

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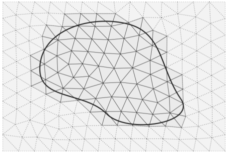

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  - stable (time) discretization
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  - linear solver concepts, ...

- suitable for strong deform./ topo. changes
- mesh generation "for free"

$\rightsquigarrow$ Assumption: we are convinced that going unfitted is an interesting idea.

How to make it work?

Section 1

Motivation

Subsection 1.2

Time Integration for unfitted FEM and moving domains: Why is time integration an issue?

Time discretization for problems on evolving geometries

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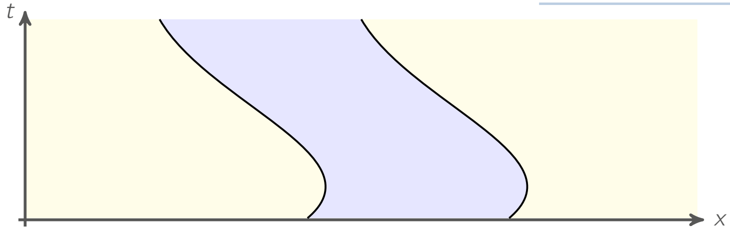


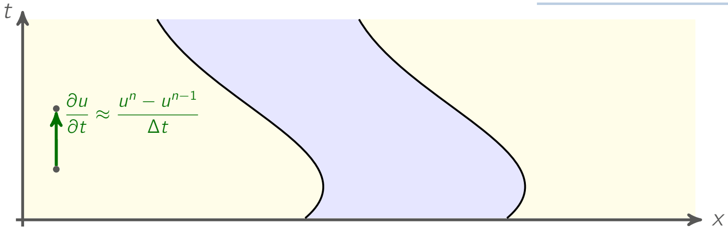
[2] P. Hansbo, M. Larson, S. Zahedi. Characteristic Cut FEM for Convection-Diffusion Problems on Time Dependent Surfaces, CMAME, 2015

[3] Q.Z. Chuwen Ma, W, Zhen. A high-order fictitious-domain method for the adv.-diff. eq. on time-varying dom., arXiv:2104.01870, 2021

[4] C.L., A. Reusken, Analysis of a Nitsche XFEM-DG discretization for a class of two-phase mass transport problems, SINUM, 2013

[5] C.L., M. Olshanskii. An Eulerian finite element method for PDEs in time-dependent domains. ESAIM: M2AN, 2019

Time discretization for problems on evolving geometries



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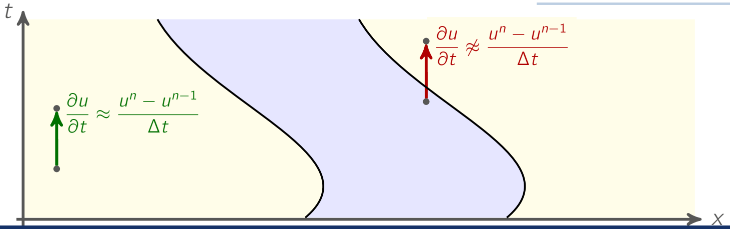

Naive method of lines is not applicable!

Alternatives:

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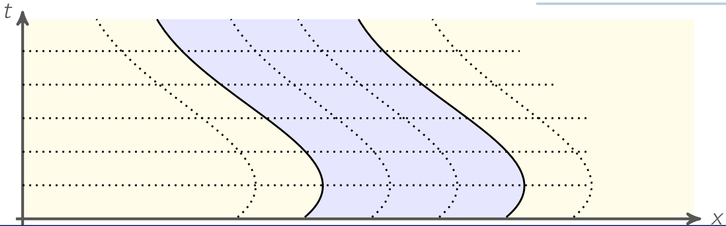

Naive method of lines is not applicable! **Alternatives:**

- Let mesh follow the geometry (Lagrangian view point)

✗ framework

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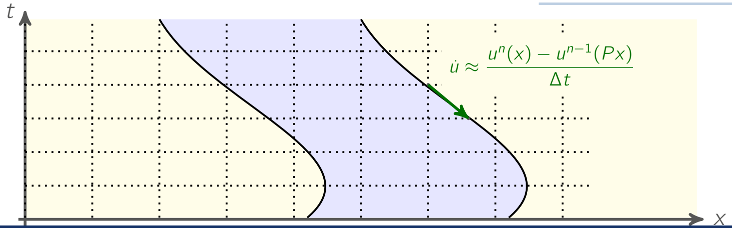

Naive method of lines is not applicable! unfitted Alternatives:

- Let mesh follow the geometry (Lagrangian view point) ✗ framework
- Use *characteristics* $\dot{u} = \frac{\partial u}{\partial t} + \mathrm{w} \cdot \nabla u$ [2,3] (✗) (framework)

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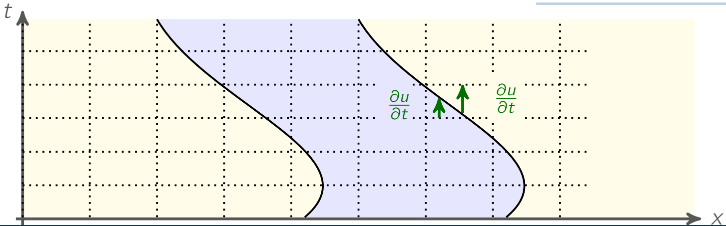

Naive method of lines is not applicable! unfitted Alternatives:

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- Use *characteristics* $\dot{u} = \frac{\partial u}{\partial t} + \mathrm{w} \cdot \nabla u$ [2,3] — (✗) (framework)
- Separate domains via a *space-time* formulation[4] — ✓

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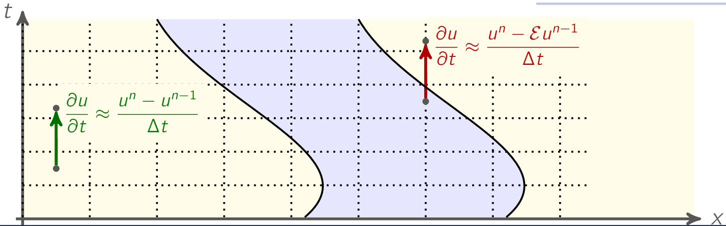

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- Separate domains via a *space-time* formulation[4] — ✓
- Extend solutions to neighborhood[5] — ✓

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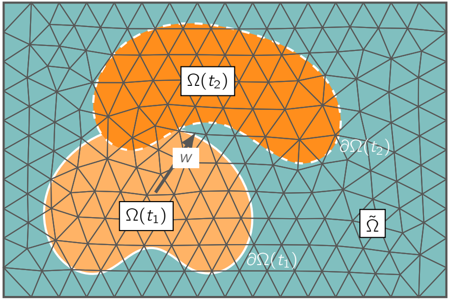
Model problem for the remainder (single phase)



$$
\begin{aligned}
\partial_t u - \nu \Delta u + \mathbf{w} \cdot \nabla u &= f && \text{in } \Omega(t), \quad t \in [0,T], \\
\nabla u \cdot \mathbf{n}_{\partial\Omega} &= 0 && \text{on } \partial\Omega(t), \quad t \in [0,T], \\
u(\cdot, t=0) &= u_0 && \text{in } \Omega(t=0).
\end{aligned}
$$

Assumptions

- $\operatorname{div}(\mathbf{w}) = 0$, $\mathbf{w} \cdot n = \mathcal{V}_n$ (no relative convective flux)
- natural boundary conditions
- $\nu = 1 > \|\mathbf{w}\|_\infty \cdot h$ (diffusion domination)

Section 2

Unfitted Space-Time FEM

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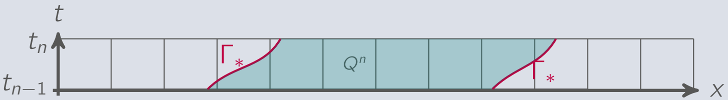
Unfitted Space-Time FEM

Subsection 2.1

Discontinuous Galerkin (in time) formulation (and variants)

Space-time FE spaces using tensor product structure (for interior domain)



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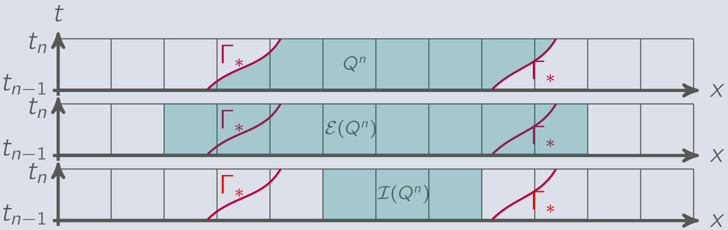
Space-time FE spaces using tensor product structure (for interior domain)



- Space-time prisms $Q^n_T = T \times I_n$ for $T \in \tilde{\mathcal{T}}_h$ (“active mesh”);
  Extended TP time slab: $\mathcal{E}(Q^n)$, included TP time slab: $\mathcal{I}(Q^n)$
- Time slab FE space (global FE space discontinuous-in-time):

$$W_n := V_h^{k_s}(\mathcal{E}(\Omega^n)) \otimes \mathcal{P}^{k_t}((t_{n-1}, t_n))$$

[6] C.L., A. Reusken. Analysis of a Nitsche XFEM-DG discretization for a class of two-phase mass transport problems. SINUM, 2013.

Illustration: three time slabs

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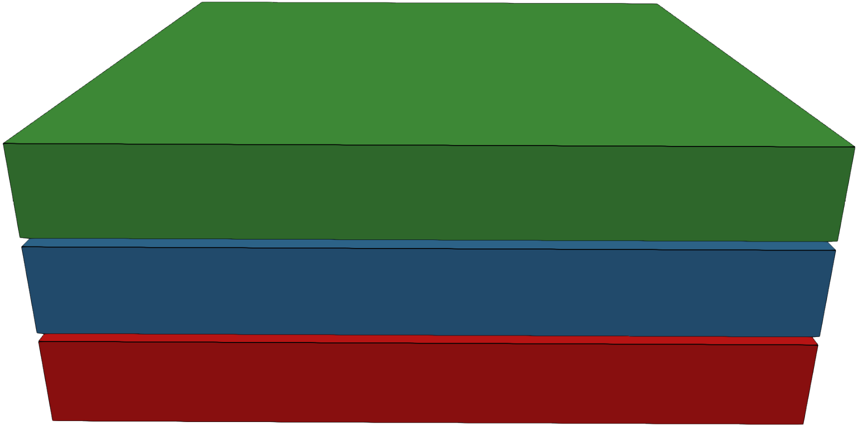
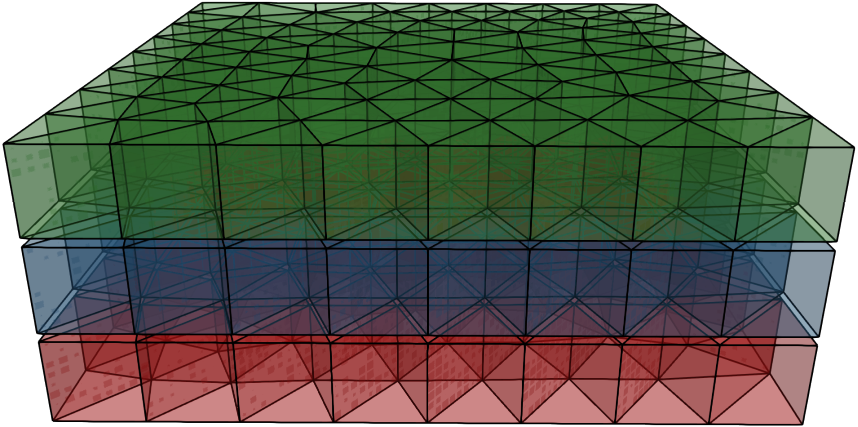


Space-Time slabs

Illustration: three time slabs



Space-Time slabs (tensor product mesh)

Illustration: three time slabs and an unfitted geometry

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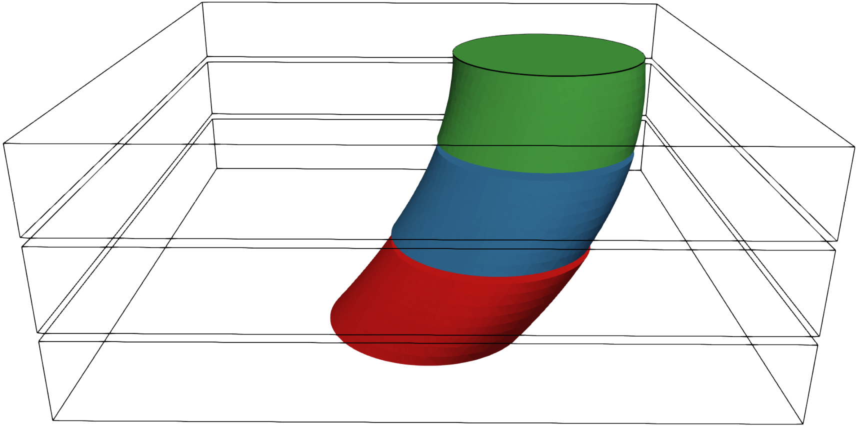
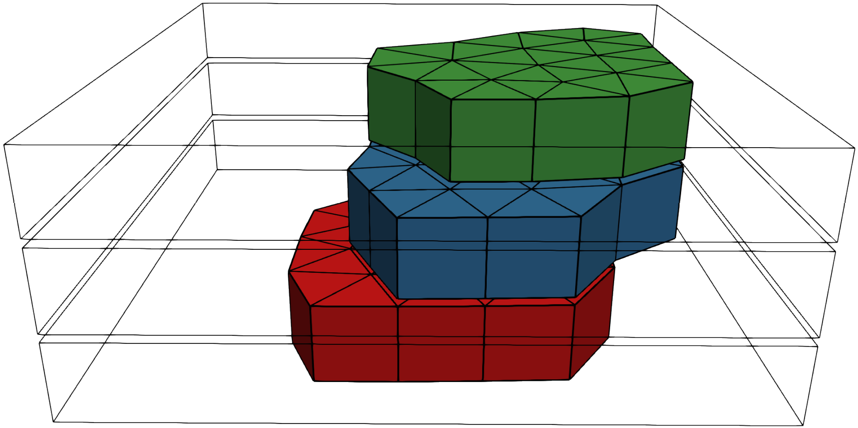


Space-Time level set domain $\phi < 0$

Illustration: three time slabs and an unfitted geometry



Active Space-Time mesh

The unfitted space-time DG formulation

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Variational formulation on each time slab (time stepping structure)

Find $u_h \in W_n$ such that for all $v_h \in W_n$ holds:

$$(\partial_t u_h + \mathbf{w} \cdot \nabla u_h, v_h)_{Q^n} + (\nabla u_h, \nabla v_h)_{Q^n} = (f, v_h)_{Q^n} \quad .$$

Contributions

- Consistency to PDE

$$\begin{aligned} \partial_t u - \Delta u + \mathbf{w} \cdot \nabla u &= f && \text{in } \Omega(t), && t \in [0, T], \\ \nabla u \cdot \mathbf{n}_{\partial\Omega} &= 0 && \text{on } \partial\Omega(t), && t \in [0, T], \end{aligned}$$

The unfitted space-time DG formulation

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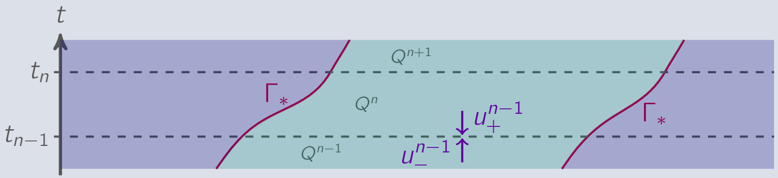
Contributions

- Consistency to PDE



- Upwind stabilization

The unfitted space-time DG formulation

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Variational formulation on each time slab (time stepping structure)

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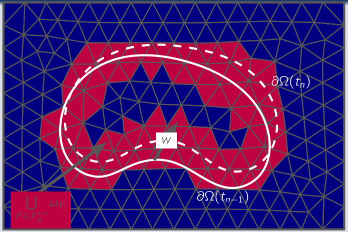
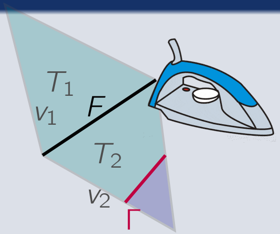
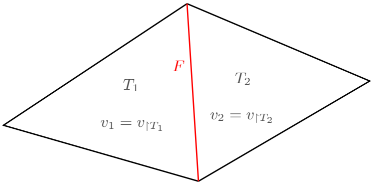
Contributions

- Consistency to PDE
- Upwind stabilization
- Ghost-penalty stabilization
  - facet-based stabilization (in the vicinity of cut prisms)
  - "glues" polynomials together
  - re-enables inverse inequalities





Space-time Ghost-penalty stabilization



Define $[\![v]\!]_{\omega_F}(x) := v_1(x) - v_2(x)$ for x in $\omega_F := T_1 \cup T_2$.
There holds

$$\|v\|_{T_1}^2 \leq C \left(\|[\![v]\!]_{\omega_F}\|_{T_1}^2 + \|v\|_{T_2}^2 \right)$$

[7] J. Preuß. Higher order unfitted isoparametric space-time FEM on moving domains. Master's thesis, 2018

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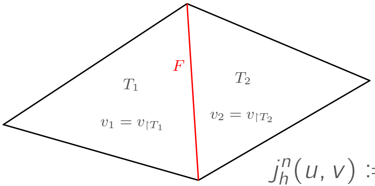

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$$j_h^n(u,v) := \int_{t_{n-1}}^{t_n} \gamma_J(\Delta t, h) \sum_{F \in \mathcal{F}_R^{*,n}} \int_{\omega_F} \frac{1}{h^2} [\![u]\!]_{\omega_F} [\![v]\!]_{\omega_F} \, dx \, dt,$$

[7] J. Preuß. Higher order unfitted isoparametric space-time FEM on moving domains. Master's thesis, 2018

Space-time Ghost-penalty stabilization

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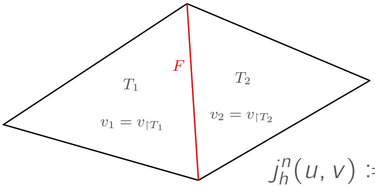

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Stabilization factor γ_J

- Stabilization factor $\gamma_J(\Delta t, h)$ may depend on anisotropic choice of Δt and h.
- γ_J scales with number of elements to cross from "bad" to "good" element
- Question: Do you need to pass
  from space-time element to space-time element ($\rightsquigarrow \gamma_J(\Delta t, h) \gtrsim (1 + \frac{\Delta t}{h})$) or
  from space element to space element for a fixed time ($\rightsquigarrow \gamma_J(\Delta t, h) \gtrsim 1$; e.g. Nitsche)?

[7] J. Preuß. Higher order unfitted isoparametric space-time FEM on moving domains. Master's thesis, 2018

Petrov-Galerkin with continuous ansatz (sketch)

- **Idea**: Instead of weakly enforcing continuity along time slices, strongly set according dofs.

Variational formulation on each time slab (C^0-continuity)

Find $u_h^n \in W_n^{k_s,k_t} \cap \{u_h^n|_{t_{n-1}} = u_h^{n-1}|_{t_{n-1}}\}$ such that for all $v_h \in W_n^{k_s,k_t-1}$ holds:

$$(\partial_t u_h^n + \mathbf{w} \cdot \nabla u_h^n, v_h)_{Q^n} + (\nabla u_h^n, \nabla v_h)_{Q^n} + j_h^n(u_h^n, v_h) = (f, v_h)_{Q^n}.$$

[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Petrov-Galerkin with continuous ansatz (sketch)

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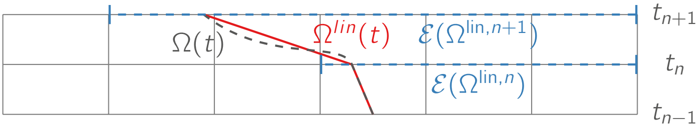
- **Problem**: Domains might not be defined accordingly (how to set initial values?).



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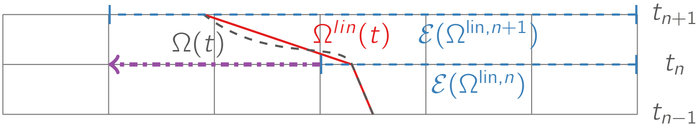

- **Idea**: Use extension (based on Ghost penalties) to obtain reasonable initial values

[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Section 2

Unfitted Space-Time FEM

Subsection 2.2

Geometry handling

Basic concept for stationary case

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Level set method

- smooth level set function ϕ
- $\Omega = \{x \in \tilde{\Omega} \mid \phi(x) < 0\}$, $\partial\Omega = \{x \in \tilde{\Omega} \mid \phi(x) = 0\}$

How to compute $\int_\Omega f \, dx$ over implicitly defined domain: Low-order approach

- approximate ϕ by piecewise linear level set function $\hat{\phi}_h = I_1 \phi$, where I_1 nodal interpolation
- $\Omega^{\text{lin}} = \{x \in \tilde{\Omega} \mid \hat{\phi}_h(x) < 0\}$, $\Gamma^{\text{lin}} = \{x \in \tilde{\Omega} \mid \hat{\phi}_h(x) = 0\}$
- $\rightsquigarrow \int\limits_\Omega f \, dx \approx \int\limits_{\Omega^{\text{lin}}} f \, dx$
- only 2^{nd} order accurate but robust

(Iso)parametric mapping for geometry approximation

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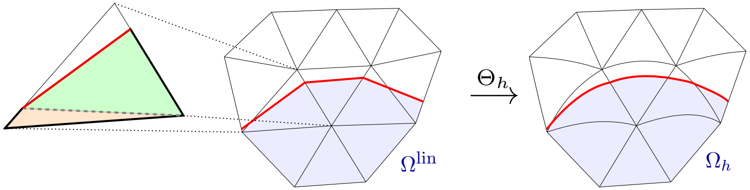
- Assume $\phi_h \in V_h^{k_s}$ high order approximation of ϕ, and $\phi^{\text{lin}} \in V_h^1$
- On $\Omega^{\text{lin}} = (\phi^{\text{lin}})^{-1}((-\infty, 0])$, numerical integration can be performed by tesselation.
- Goal: Transfer these geometries by isoparametric mapping $\Theta_h : \tilde{\Omega} \to \tilde{\Omega}, \Theta_h \in (V_h^{k_s})^d$.



- Integration only needs to deal with Ω^{lin} (deformation Θ_h changes integrands)
- limited cut topologies (feature and bug)

[9] C.L., High order unfitted finite element methods on level set domains using isoparametric mappings. CMAME, 2016.

(Iso)parametric mapping for space-time case

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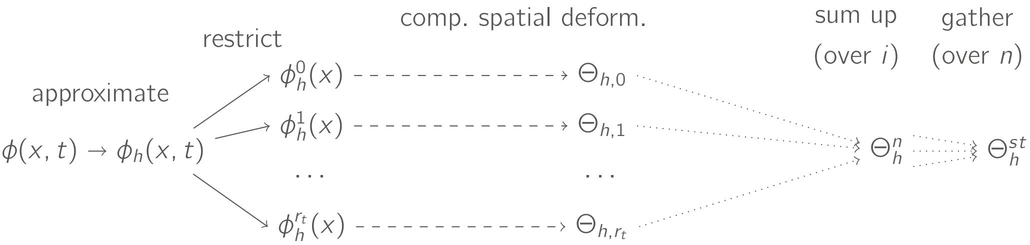
Assume $\phi_h \in V_h^{q_s} \otimes \mathcal{P}^{q_t}(I_n)$ and define $\Theta_h^n(x,t) = \sum_{i=0}^{r_t} \ell_i(t) \cdot \Theta_{h,i}$

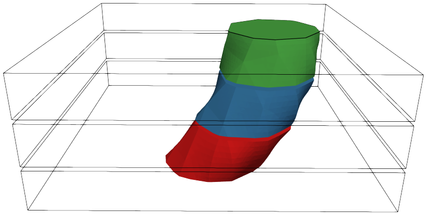


$$\Omega^h(t) := \Theta_h^{\mathrm{st}}(\Omega^{\mathrm{lin}} t, t), \quad Q^{\mathrm{lin},n} = \bigcup_{t \in I_n} \Omega^{\mathrm{lin}}(t) \times \{t\}, \quad Q^{h,n} = \Theta_h^{\mathrm{st}}(Q^{\mathrm{lin},n}).$$

[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Space-time reference conf. and isoparametric mapping

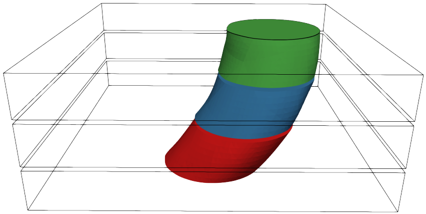
$\{\hat{\phi}_h(x,t)=0\},\ \hat{\phi}_h(x,t)\in V_h^1\otimes\mathcal{P}^{q_t}$



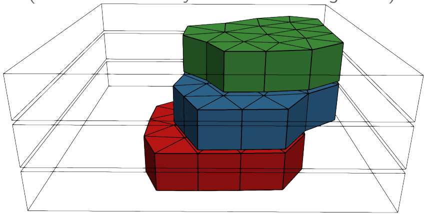
piecewise linear-in-space zero level set
(allows for arbitrary order num. integration)

$\xrightarrow{\Theta_h}$

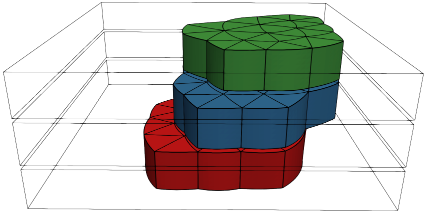
$\Theta_h(\{\hat{\phi}_h(x,t)=0\}),\ \Theta_h(t):\tilde{\Omega}\to\tilde{\Omega}\in[V_h^{q_s}]^d\otimes\mathcal{P}^{q_t}$



explicit space-time level set domain



active space-time mesh $\mathcal{E}(Q^n)$



mapped mesh

Section 3

A priori error analyses of DG (in time) method

Section 3

A priori error analyses of DG (in time) method

Subsection 3.1

Preparations:
Integration by parts and the Ghost penalty mechanism

Preparations

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Formulation on the whole space-time domain

Find $u \in W_h$ s.t. for all $v \in W_h$ $B(u,v) + J(u,v) = f(v)$ with

$$B(u,v) := \underbrace{\sum_{n=1}^{N} (\partial_t u + \mathbf{w} \cdot \nabla u, v)_{Q^n}}_{=:d(u,v)} + \underbrace{\sum_{n=1}^{N} (\nabla u, \nabla v)_{Q^n}}_{=:a(u,v)} + \underbrace{\sum_{n=1}^{N-1} \left([\![u]\!]^n, v_+^n\right)_{\Omega^n} + \left(u_+^0, v_+^0\right)_{\Omega^0}}_{=:b(u,v)},$$

$$J(u,v) := \sum_{n=1}^{N} j_h^n(u,v), \quad f(v) := \sum_{n=1}^{N} (f,v)_{Q^n} + \left(u_0, v_+^0\right)_{\Omega^0}, \quad \text{where } [\![u]\!]^n = u_+^n - u_-^n.$$

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Assumptions

- Exact geometry handling, smooth domains
- shape-regular, quasi-uniform background mesh
- constant time step $\Delta t_n \equiv \Delta t$

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Lemma

For $u, v \in W_h + H^1(Q)$ *there holds*

$$\underbrace{\sum_{n=1}^{N} (\partial_t u + \mathbf{w} \cdot \nabla u, v)_{Q^n}}_{=:d(u,v)} + \underbrace{\sum_{n=1}^{N-1} \left([\![u]\!]^n, v_+^n \right)_{\Omega^n} + \left(u_+^0, v_+^0 \right)_{\Omega^0}}_{=:b(u,v)} = d'(u,v) + b'(u,v) \quad \textit{with}$$

$$d'(u,v) := \sum_{n=1}^{N} (u, -(\partial_t v + \mathbf{w} \cdot \nabla v))_{Q^n}, \qquad b'(u,v) := -\sum_{n=1}^{N-1} \left(u_-^n, [\![v]\!]^n \right)_{\Omega^n} + \left(u_-^N, v_-^N \right)_{\Omega^N}.$$

Corollary ($v = u$)

$$(d+b)(u,u) = \left(\tfrac{d+d'}{2} + \tfrac{b+b'}{2}\right)(u,u) = \tfrac{1}{2} [\![u]\!]^2 \;\; \textit{with} \;\; [\![u]\!]^2 := \sum_{n=1}^{N-1} \left([\![u]\!]^n, [\![u]\!]^n \right)_{\Omega^n} + \left(u_+^0, u_+^0 \right)_{\Omega^0} + \left(u_-^N, u_-^N \right)_{\Omega^N}.$$

Ghost penalty control

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Lemma

There exists a constant $C > 0$ such that for every $u \in W_h \oplus \nabla W_h$ there holds

$$\|u\|^2_{\mathcal{E}(Q^n)} \leq C \left(\frac{h^2}{\gamma_J} j_h^n(u,u) + \|u\|^2_{\mathcal{I}(Q^n)} \right) \text{ and } \|u\|^2_{\mathcal{E}(Q)} \leq C \left(\frac{h^2}{\gamma_J} J(u,u) + \|u\|^2_{\mathcal{I}(Q)} \right).$$

Key result for analysis

- bound norm of discrete function on $\mathcal{E}(Q^n)$ by its norm on $\mathcal{I}(Q^n)$ (plus stabilization terms)
  [directly clear for L^2 norm, but can be extended to other relevant norms]
- allows to extend estimates for finite elements with tensor product structure to unfitted case

Section 3

A priori error analyses of DG (in time) method

Subsection 3.2

Coercivity-based analyses

A coercivity result

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Norms

$$\|u\|_j^2 := \|u\|^2 + \|u\|_J^2, \qquad \|u\|^2 := [\![u]\!]^2 + \sum_{n=1}^{N} \|\nabla u\|_{Q^n},$$

$$[\![u]\!]^2 := \sum_{n=1}^{N-1} ([\![u]\!]^n, [\![u]\!]^n)_{\Omega^n}, \qquad \|u\|_J^2 := J(u,u).$$

Lemma

For all $u \in W_h$ there holds

$$B(u,u)(+J(u,u)) \geq \frac{1}{2}[\![u]\!]^2 + \sum_n \|\nabla u\|_{Q^n}^2 (+\|u\|_J^2) \gtrsim \|u\|_j^2$$

Note: We do not require Ghost penalty stabilization (i.e. $\gamma_J > 0$) here.

An **unfinished** Céa-type result (no Ghost Penalty)

Lemma (Céa Lemma)

There holds the a priori error estimate $\|u - u_h\| \lesssim \|u - u_I\|_*$ *with* $\|\cdot\|_* =?$ *and* $u_I =?$

An unfinished Céa-type result (no Ghost Penalty)

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Lemma (Céa Lemma)

There holds the a priori error estimate $\|u - u_h\| \lesssim \|u - u_I\|_*$ *with* $\| \cdot \|_* =?$ *and* $u_I =?$

Proof.

$$\|u_I - u_h\|^2 \lesssim B(u_I - u_h, u_I - u_h) \overset{\text{Gal. orth.}}{=} B(u_I - u, u_I - u_h) \overset{!}{\lesssim} \|u - u_I\|_* \|u_h - u_I\|$$

$$+\|u_I - u_h\|_J^2 \quad +J(u_I - u_h, u_I - u_h) \quad +J(u_I, u_I - u_h) \quad +\|u_I\|_J \|u_h - u_I\|_J$$

$$\Rightarrow \|u_I - u_h\| \lesssim \|u - u_I\|_*$$

$$\Rightarrow \|u - u_h\| \leq \|u_I - u_h\| + \|u_I - u\| \lesssim \|u - u_I\|_* \qquad \square$$

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$$\Rightarrow \|u_I - u_h\| \lesssim \|u - u_I\|_*$$

$$\Rightarrow \|u - u_h\| \leq \|u_I - u_h\| + \|u_I - u\| \lesssim \|u - u_I\|_*$$

□

Conclusion:

We need $B(u_I - u, u_I - u_h) \lesssim \|u - u_I\|_* \|u_h - u_I\|$ and have essentially two degrees of freedom:

Interpolant u_I and norm $\|\cdot\|_*$

Approach 1

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- Method and analysis for moving interface problem with special Nitsche for interface conditions (no Ghost penalties)
- Use continuous (in-time) interpolant u_I, s.t. $u - u_I$ is continuous in time s.t. $[\![\cdot]\!]^n$ terms vanish.
- Choose $|\!|\!| \cdot |\!|\!|_* = \| \cdot \|_{H^1(Q)}$ (space-time H^1 norm), i.e. $\partial_t(u - u_I)$ can be controlled
- Consequence:

$$|\!|\!| u - u_h |\!|\!| \lesssim |\!|\!| u - u_I |\!|\!|_* = \| u - u_I \|_{H^1(Q)}$$

- Suboptimal in time by one order (anisotropic case)

[6] C.L., A. Reusken. Analysis of a Nitsche XFEM-DG discretization for a class of two-phase mass transport problems. SINUM, 2013.

Approach 2

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- Construct interpolant $u_I = \mathcal{I}_Q u$ so that for all n

$$\int_{Q^n} (u - \mathcal{I}_Q u) w_h \, d(x,t) = 0 \;\forall w_h \in V_h^{k_s,n} \otimes \mathcal{P}^{k_t-1}(I_n) = \partial_t W_h, \quad \int_{\Omega^n} (u - \mathcal{I}_Q u) w_h \, dx = 0 \;\forall w_h \in V_h^{k_s,n}$$

- Apply partial integration and set $\|\!|\cdot|\!\|_* = \|\!|\cdot|\!\|$

$$(d+b)(u_I - u, u_I - u_h) = (d' + b')(u_I - u, u_I - u_h)$$
$$\overset{u_I = \mathcal{I}_Q u}{=} \underbrace{d'(\mathcal{I}_Q u - u, w_h)}_{\partial_t \ldots \rightsquigarrow \not{\partial}_t \ldots} + \underbrace{b'(\mathcal{I}_Q u - u, w_h)}_{=0} \lesssim \|\!|\mathcal{I}_Q u - u|\!\| \, \|\!|w_h|\!\|$$

- Price to pay: Rely on $\mathcal{I}_Q^{\mathrm{TP}}$: $\|\nabla(u - \mathcal{I}_Q u)\|_{Q^n} \leq \|\nabla(u - \mathcal{I}_Q^{\mathrm{TP}} u)\|_{Q^n} + \underbrace{\|\nabla(\mathcal{I}_Q u - \mathcal{I}_Q \mathcal{I}_Q^{\mathrm{TP}} u)\|_{Q^n}}_{\lesssim h^{-1} \|u - \mathcal{I}_Q^{\mathrm{TP}} u\|_{Q^n}}$

- Error bound: $\|\!|u - u_h|\!\| \lesssim h^{k_s} |u|_{H^{k_s+1,0}(Q)} + h^{-1} \Delta t^{k_t+1} |u|_{H^{0,k_t+1}(Q)} + h^{k_s+1} \sup_t |u|_{H^{k_s+1}(\Omega(t))}$

[10] S. Badia, H. Dilip, F. Verdugo. Space-time aggregated FEM for time-dependent problems on moving domains. arxiv:2206.03626, 2013.

Ghost penalty

- Ghost penalties are not needed for the a priori analysis (assuming exact arithmetics)
- Ghost penalties are (only) crucial for conditioning
- Ghost penalties also don't hurt
- Both works treat problems with Dirichlet-type boundary or interface conditions with Nitsche. Then, Ghost penalties [a] become necessary[b]

[a] or alike (e.g. aggregated FEM)

[b] with a special form of Nitsche's method the interface problem can even be done without Ghost penalties.

[6] C.L., A. Reusken. Analysis of a Nitsche XFEM-DG discretization for a class of two-phase mass transport problems. SINUM, 2013.

[10] S. Badia, H. Dilip, F. Verdugo. Space-time aggregated FEM for time-dependent problems on moving domains. arxiv:2206.03626, 2013.

Section 3

A priori error analyses of DG (in time) method

Subsection 3.3

Inf-Sup based analysis

Approach 3

Motivation

- Consider problem as similar to a linear transport problem
- Space-time convection: $\partial_t + \mathbf{w} \cdot \nabla = \nabla^*$
- Treat time derivative as convection in linear transport DG analysis

[7] J. Preuß. Higher order unfitted isoparametric space-time FEM on moving domains. Master's thesis, 2018

Approach 3

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Motivation

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- Treat time derivative as convection in linear transport DG analysis

Norms

Two norms: [Δt takes the role of Δ^{-1}]

- "Stability norm": $|||u|||^2 := \sum_{n=1}^{N} (\Delta t \partial_t u, \partial_t u)_{Q^n} + [\![u]\!]^2 + \sum_{n=1}^{N} (\nabla u, \nabla u)_{Q^n}$
- "Continuity norm": $|||u|||_*^2 := \sum_{n=1}^{N} \left(\frac{1}{\Delta t} u, u\right)_{Q^n} + [\![u]\!]_*^2 + \sum_{n=1}^{N} (\nabla u, \nabla u)_{Q^n}$ with $[\![u]\!]_*^2 := \sum_{n=1}^{N} \left(u_-^n, u_-^n\right)_{\Omega^n}$.
- Pairing is tailored for continuity while balancing approximation in time: $B(u, v) \lesssim |||u|||_* |||v|||$
- No restriction on interpolation operator

[7] J. Preuß. Higher order unfitted isoparametric space-time FEM on moving domains. Master's thesis, 2018

Inf-Sup stability

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Lemma

For all $u \in W_h$ there is a $v \in W_h$ so that there holds

$$B(u,v) + J(u,v) \gtrsim |||u|||_j |||v|||_j$$

Sketch of proof.

- Ghost penalty allows to reduce the problem to problem on interior tensor-product domain $\mathcal{I}(Q)$
- For $u \in W_h$ there holds $v^* = \Delta t \partial_t u \in W_h$

$$\rightsquigarrow (\partial_t u, v^*)_{Q^n} = (\partial_t u, \Delta t \partial_t u)_{Q^n} \quad \text{and} \quad |||v^*|||_j \simeq |||u|||_j$$

- Set $v = u + \alpha v^*$
- + technicalities

□

[7] J. Preuß. Higher order unfitted isoparametric space-time FEM on moving domains. Master's thesis, 2018

Theorem

Let $k_{\max} = \max\{k_s, k_t\}$. *Then there holds:*

$$|||u - u_h||| \lesssim \left(\Delta t^{k_t+1/2} + \sqrt{\left(1 + \frac{\Delta t}{h}\right)} h^{k_s} \right) \|u\|_{H^{k_{\max}+2}(Q)},$$

(semi-)norm $\lvert\cdot\rvert = \dots$	approximation error $\inf_{w_h \in W_h} \lvert u - w_h \rvert \leq C \cdot \dots$	discretization error $\lvert u - u_h \rvert \leq C \cdot \dots$
$\Delta t^{\frac{1}{2}} \|\partial_t \cdot\|_Q$	$\Delta t^{k_t+\frac{1}{2}} + h^{k_s+1}$	$\Delta t^{k_t+\frac{1}{2}} + \sqrt{\left(1 + \frac{\Delta t}{h}\right)} h^{k_s}$
$\|\nabla \cdot\|_Q$	$\Delta t^{k_t+1} + h^{k_s}$	$\Delta t^{k_t+\frac{1}{2}} + \sqrt{\left(1 + \frac{\Delta t}{h}\right)} h^{k_s}$

The result is optimal in the chosen (stronger) norm (up to the space-time anisotropy factor).

[7] J. Preuß. Higher order unfitted isoparametric space-time FEM on moving domains. Master's thesis, 2018

Comparison of error bounds

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Error bounds

Approach 1: $\|u - u_h\| \lesssim h^{k_s} |u|_{H^{k_s+1,0}(Q)} + \Delta t^{k_t} |u|_{H^{0,k_t+1}(Q)}$

Approach 2: $\|u - u_h\| \lesssim h^{k_s} |u|_{H^{k_s+1,0}(Q)} + h^{-1} \Delta t^{k_t+1} |u|_{H^{0,k_t}(Q)} + h^{k_s+1} \sup_t |u|_{H^{k_s+1}(\Omega(t))}$

Approach 3: $|||u - u_h||| \lesssim \left(\sqrt{\left(1 + \frac{\Delta t}{h}\right)} h^{k_s} + \Delta t^{k_t+1/2} \right) \|u\|_{H^{k\max+2}(Q)}$

Remarks

- Anisotropy factor does not appear for Approaches 1,2 as Ghost Penalty was not needed.
- Strong regularity assumption in Approach 3 can be reduced (simplification).
- Isotropic case ($\Delta t \simeq h$) and equal (or lower spatial) order $k_s \leq k_t$: same rates (h^{k_s})
- Isotropic case ($\Delta t \simeq h$), higher spatial order $k_s \geq k_t$: Approach 1,2: Δt^{k_t}, Approach 3: $\Delta t^{k_t + \frac{1}{2}}$
- Duality techniques not obvious for any of these cases

  [for Approach 1 $H^{-1}(\Omega(T))$-norm bounds exist]

Section 4

Numerical examples

Kite geometry

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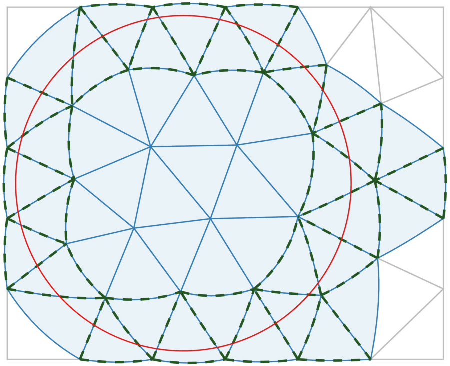
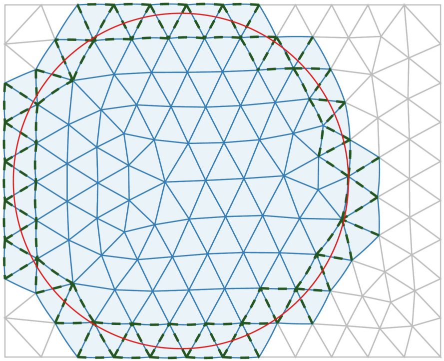
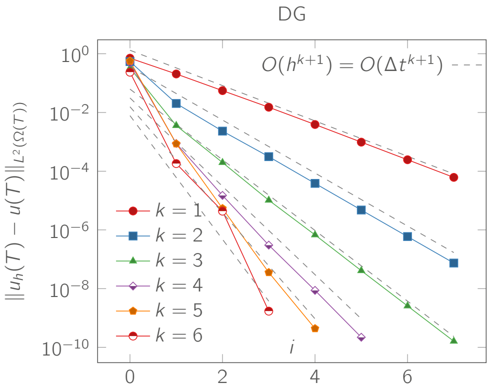


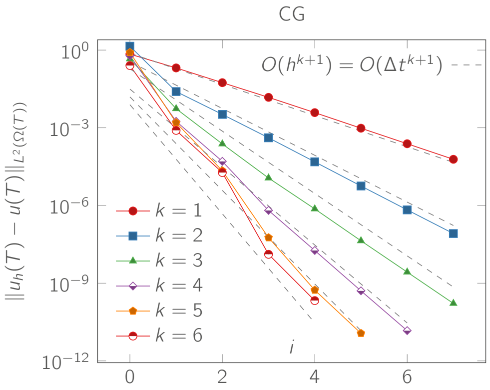


[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Convergence with $k_t = k_s$ for DG/CG

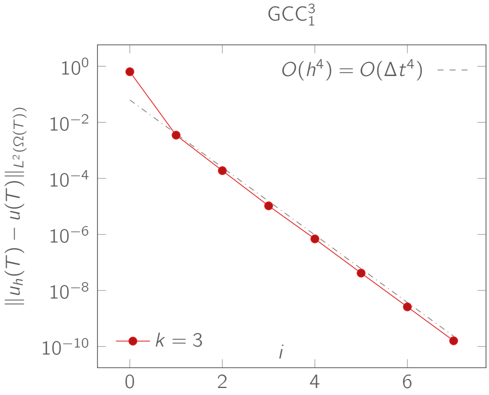
First setup: Kite with $k = k_s = k_t = q_s = q_t$, $i = i_s = i_t$, manufactured f.





[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Convergence with $k_t = k_s$ for GCC

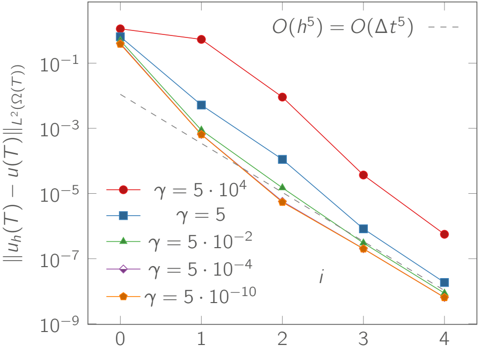


Overall, we observe

$$\|u - u_h\|_{L^2(\Omega(T))} + \|u - u_h\|_{L^2(L^2(\Omega(t)),0,T)} = \mathcal{O}(h^{k+1}) = \mathcal{O}(\Delta t^{k+1}).$$

[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Choice of the stabilisation parameter



- Consider DG, $k = 4$

$\Rightarrow$ No significant impact of stab. parameter.

[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Superconvergence

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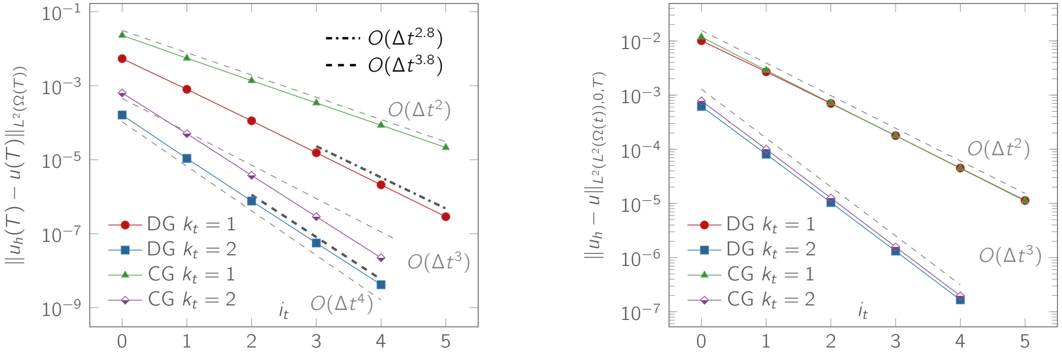
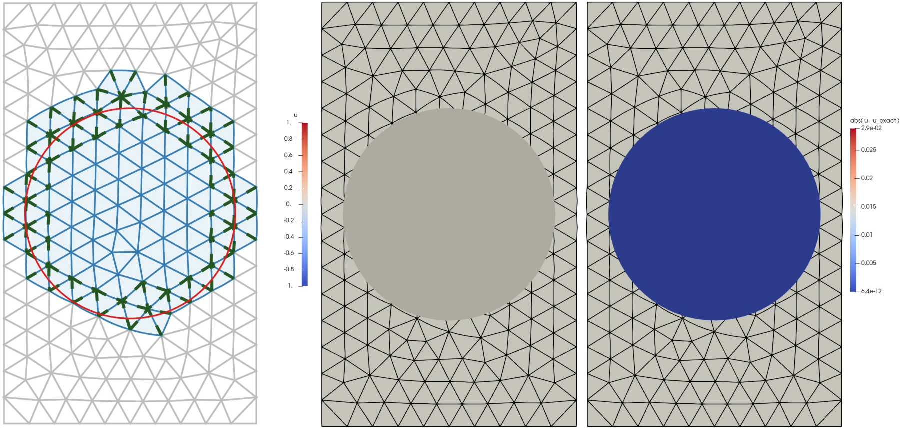
Idea: Investigate $(k_t, k_s) = (1,3), (2,5)$ to see whether we obtain more than $\mathcal{O}(\Delta t^{k_t+1})$.



Result: $\mathcal{O}(\Delta t^{k_t+1.8})$ for DG. For CG no unique result.

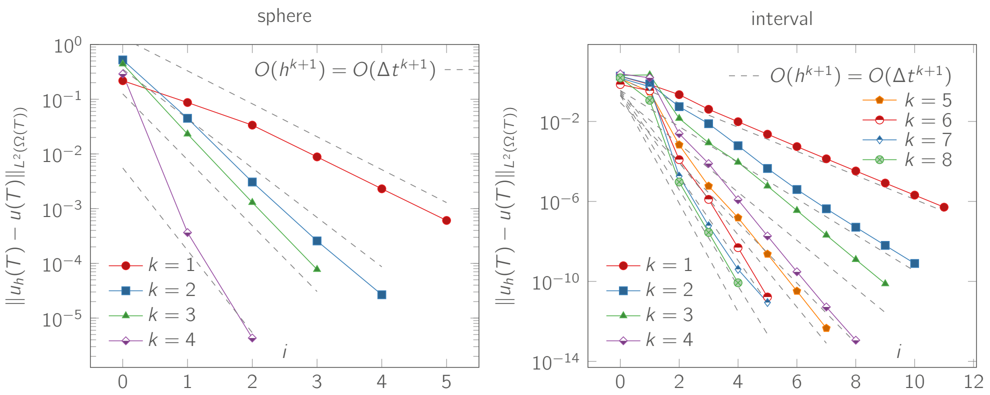
[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Moving n-sphere



[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Moving sphere / inverval, $k_s = k_t = k$



[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Non-zero entries for 2D example

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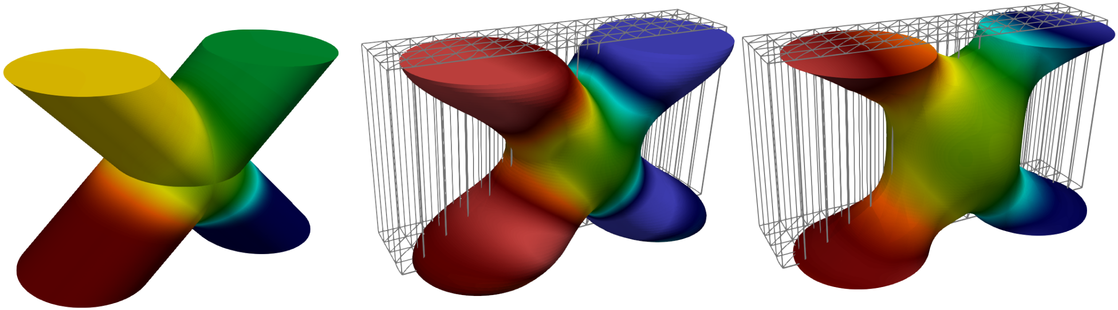
	$i_t = 1$		$i_t = 3$		$i_t = 5$	
	nzes	err	nzes	err	nzes	err
DG($k=1$):	3.57K – 4.24K	$(1.3\cdot10^{-1})$	3.3K - 3.31K	$(3.4\cdot10^{-2})$	2.79K – 3.14K	$(3.3\cdot10^{-2})$
CG($k=1$):	1.63K – 1.72K	$(1.7\cdot10^{-1})$	978 - 1.6K	$(3.3\cdot10^{-2})$	728 – 867	$(3.3\cdot10^{-2})$
DG($k=3$):	353K – 442K	$(6.6\cdot10^{-4})$	293K - 327K	$(2.6\cdot10^{-4})$	267K – 300K	$(2.7\cdot10^{-4})$
CG($k=3$):	221K – 268K	$(1.6\cdot10^{-3})$	172K - 190K	$(2.9\cdot10^{-4})$	150K – 171K	$(2.7\cdot10^{-4})$
GCC($k=3$):	145K – 151K	$(6.1\cdot10^{-3})$	84.5K - 92.2K	$(3.1\cdot10^{-4})$	62.9K – 76.1K	$(2.8\cdot10^{-4})$
DG($k=5$):	4.39M –5.58M	$(5.4\cdot10^{-6})$	3.60M - 4.5M	$(2.8\cdot10^{-7})$	3.26M –3.69M	$(2.2\cdot10^{-7})$
CG($k=5$):	3.71M –4.43M	$(1.7\cdot10^{-5})$	2.69M - 2.99M	$(1.2\cdot10^{-6})$	2.28M –2.63M	$(4.3\cdot10^{-7})$

Range for non-zero entries (nzes) of system matrix (depends on time step) and absolute $L^2(T)$-error (err) in comparison for DG, CG and GCC. In this whole table, $i_s = 2$.

[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Colliding circles

- A test case in 2D with topology change:
- Two circles with different constant concentration merge and seperate.
- 1 time step of DG method:



[8] F. Heimann, C.L., J. Preuß. Geometrically higher order unfitted space-time methods for pdes on moving domains. arXiv:2202.02216, accepted (SISC), 2022.

Section 5

Summary and outlook

Summary and outlook

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Summary

- Arbitrary high order and robust unfitted methods for moving domains
- higher order geometry handling (numerically tested)
- a priori error bounds (for exact geometry handling)

Issues / ongoing work

- Improved a priori error analysis
- A priori error analysis for continuous-in-time variants
- A priori error analysis including geometry approximation
  - Geometry errors perturb some important structural results
    [integration by parts, ..]
  - Parametric mapping leads to difficulties
    [discontinuous-in-time mapping $\rightsquigarrow$ mesh transfer operations, ..]
- Applications beyond the scalar model problem

Summary and outlook

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    [discontinuous-in-time mapping $\rightsquigarrow$ mesh transfer operations, ..]
- Applications beyond the scalar model problem

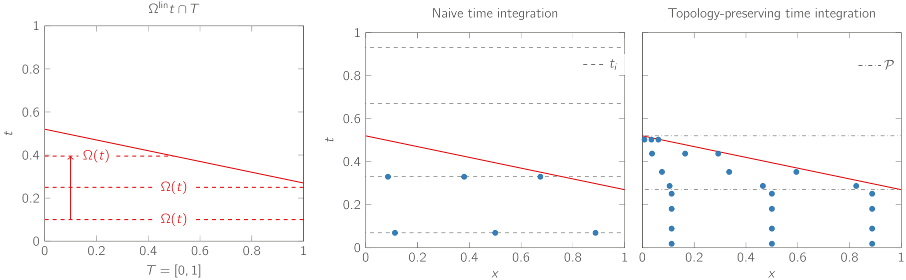
Thank you for your attention!

Section 6

Appendix

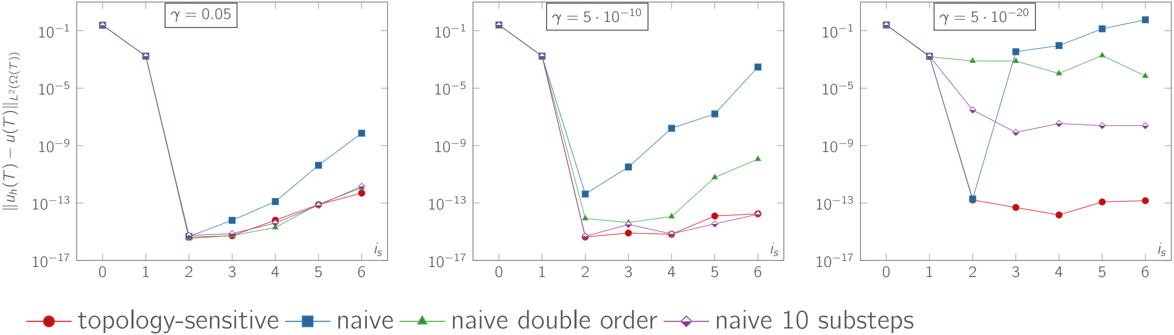
Topology-preserving time integration

- General strategy: iterated integrals
- On each prism the reference level set function is $\mathcal{P}^1(T) \times \mathcal{P}^{q_t}(I_n)$.
- Integrand becomes nonsmooth where level set function touches (space) vertices
- Decompose outer integration (time) into sub-intervals



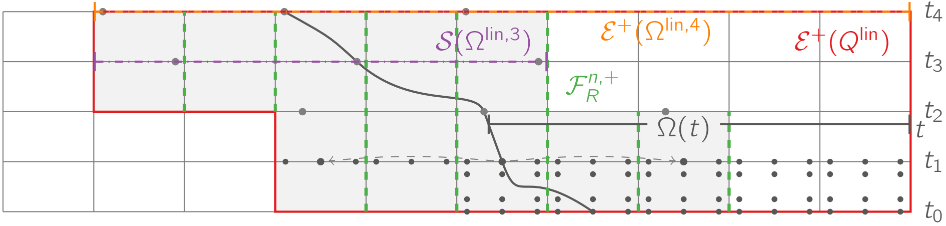
Different time integration strategies

- By and large, naive and topology-sensitive time integration yield similar results.
- Differences in 1D example, $k = 4$, $i_t = 0$.



$\Rightarrow$ Ghost-penalty stabilisation can fix stability issues from naive time integration.

Petrov-Galerkin with continuous ansatz (sketch) II



Variational formulation (w. homogenization and extension): Find $u = u_0 + u_{\text{init}}$ with $u_0 \in W_n^{k_s,k_t} \cap \{u_h^n|_{t_{n-1}} = 0\}$ and $u_{\text{init}} \in W_n^{k_s,k_t}$ (given) s.t.

$$B^n(u_0, v) + j_h^n(u_0, v) + j_h^{n,*}(\mathcal{F}_R^{n,+}, t_n; (u_0)_-^n, v_-^n) = f^n(v) - B^n(u_{\text{init}}, v) \quad \forall v \in V_h^n,$$

- Some details are missing
- Idea extends to methods with higher regularity in time (Galerkin collocation methods).