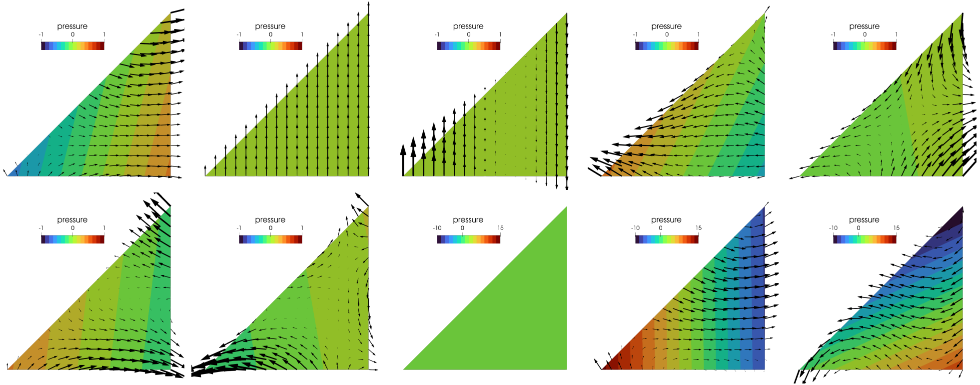
Trefftz-DG for Stokes problems

~~Christoph Lehrenfeld~~ Igor Voulis
(joint work with P. Lederer and P. Stocker)
Institute for Numerical and Applied Mathematics, Georg-August University

March 22, 2024, GAMM Annual Meeting 2024, S18



The Stokes problem

$\Omega \subset \mathbb{R}^d$ with $d = 2, 3$. Find velocity u and pressure p s.t.

$$-\nu \Delta u + \nabla p = f \quad \text{in } \Omega, \tag{Sa}$$

$$-\operatorname{div} u = g \quad \text{in } \Omega, \tag{Sb}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{Sc}$$

where f, g are ext. forces/sources and $\nu > 0$ is the dynamic viscosity. Weak formulation of (Sa)–(Sc): Find $(u, p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$, s.t.

$$\begin{aligned} \int_\Omega \nu \nabla u : \nabla v \, dx - \int_\Omega \operatorname{div} v \, p \, dx &= \int_\Omega f \cdot v \, dx \quad \forall v \in [H_0^1(\Omega)]^d, \\ - \int_\Omega \operatorname{div} u \, q \, dx &= \int_\Omega g \, q \, dx \quad \forall q \in L_0^2(\Omega), \end{aligned} \tag{W}$$

Standard DG for Stokes

Find $(u_h, p_h) \in \mathbb{X}_h^k := [\mathbb{P}^k]^d \times \mathbb{P}^{k-1}/\mathbb{R}$, s.t.

$$a_h(u_h, v_h) + b_h(v_h, p_h) = (f, v_h)_{\mathcal{T}_h} \quad \forall\, v_h \in [\mathbb{P}^k]^d, \tag{DGa}$$

$$b_h(u_h, q_h) = (g, q_h)_{\mathcal{T}_h} \quad \forall\, q_h \in \mathbb{P}^{k-1}/\mathbb{R}, \tag{DGb}$$

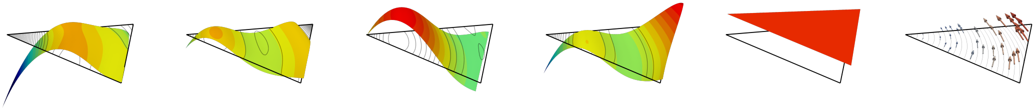
with the bilinear forms

$$\begin{aligned} a_h(u_h, v_h) &:= (\nu \nabla u_h, \nabla v_h)_{\mathcal{T}_h} - (\{\!\{\nu \partial_n u_h\}\!\}, [\![v_h]\!])_{\mathcal{F}_h} - (\{\!\{\nu \partial_n v_h\}\!\}, [\![u_h]\!])_{\mathcal{F}_h} \\ &\quad + \frac{\alpha \nu}{h}([\![u_h]\!], [\![v_h]\!])_{\mathcal{F}_h}, \\ b_h(v_h, p_h) &:= -(\operatorname{div} v_h, p_h)_{\mathcal{T}_h} + ([\![v_h \cdot n]\!], \{\!\{p_h\}\!\})_{\mathcal{F}_h}, \end{aligned}$$

where the interior penalty parameter $\alpha = \mathcal{O}(k^2)$ is chosen sufficiently large and we used the notation $\partial_n w := \nabla w \cdot n$ (and std. notation for avg. and jumps).

Idea Trefftz (in a nutshell):

- Take your DG formulation and replace the polynomial spaces by Trefftz spaces.
- Trefftz spaces are spaces of functions that satisfy the PDE in the interior exactly (not considering boundary / element interface conditions), i.e.
  - harmonic polynomials for the Laplace equation,
  - plane waves for the Helmholtz equation,
  - etc.
- Exact solutions are in general hard to find,
  but approximate solutions can be constructed (Quasi-Trefftz / Weak Trefftz)



The Stokes Trefftz DG space

$$\mathbb{T}^k_{f,g}(\mathcal{T}_h) := \{(u_h, p_h) \in \mathbb{X}^k_h \mid -\Delta u_h + \nabla p_h = \Pi^{k-2} f, -\operatorname{div} u_h = \Pi^{k-1} g\}, \tag{T}$$

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Only affine linear. Decompose into

$$\mathbb{T}^k = \mathbb{T}^k(\mathcal{T}_h) = \mathbb{T}^k_{0,0}(\mathcal{T}_h) \quad \text{and a particular solution} \quad (u_{h,p}, p_{h,p}) \in \mathbb{T}^k_{f,g}(\mathcal{T}_h).$$

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Find $(u_h, p_h) \in \mathbb{T}_{f,g} = \mathbb{T} + (u_{h,p}, p_{h,p})$ such that $\forall (v_h, q_h) \in \mathbb{T}$

$$K_h((u_h, p_h), (v_h, q_h)) := a_h(u_h, v_h) + b_h(v_h, p_h) + b_h(u_h, q_h) = (f, v_h)_{\mathcal{T}_h} + (g, q_h)_{\mathcal{T}_h}. \tag{T-DG}$$

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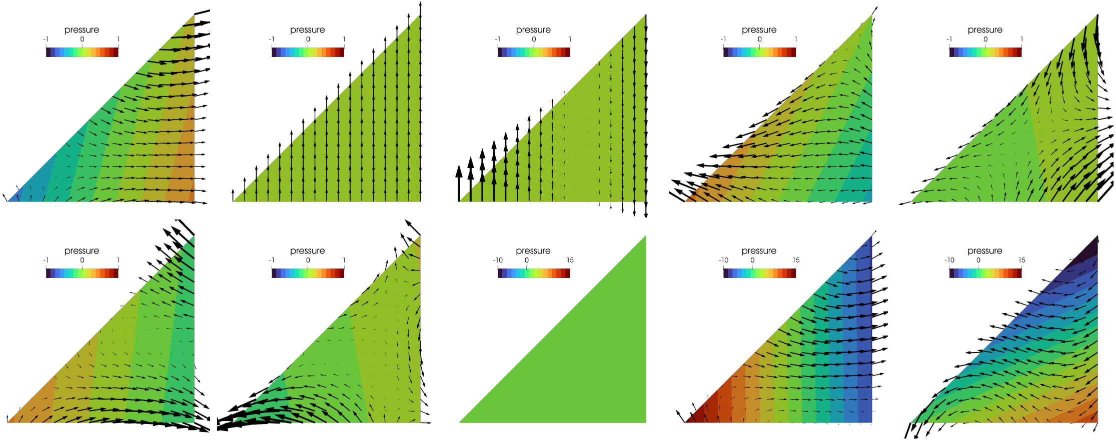
A few questions:

- Does a particular solution always exist?
- How to compute it?
- How much do we gain from $\mathbb{X}^k_h = [\mathbb{P}^k]^d \times \mathbb{P}^{k-1} \rightsquigarrow \mathbb{T}^k$?

Example basis functions ($k = 2$)



Lemma

The pointwise Stokes operator $\mathcal{L} : [\mathcal{P}^k(T)]^d \times \mathcal{P}^{k-1}(T) \to [\mathcal{P}^{k-2}(T)]^d \times \mathcal{P}^{k-1}(T)$, $(v, q) \mapsto (-\Delta v + \nabla p, -\operatorname{div} v)$ is surjective and the local Trefftz space on an element $T \in \mathcal{T}_h$ has

$$\dim(\mathbb{T}(T)) = \dim(\mathbb{X}_h(T)) - \dim([\mathcal{P}^{k-2}]^d) - \dim(\mathcal{P}^{k-1}) = d\left(\binom{k+d}{d} - \binom{k-2+d}{d}\right).$$

	$d=2$	$d=3$
$\dim \mathbb{X}_h(T)$	$\frac{3}{2}k^2 + \frac{7}{2}k + 2$	$\frac{2}{3}k^3 + \frac{7}{2}k^2 + \frac{35}{6}k + 3$
$\dim \mathbb{T}\ (T)$	$4k + 2$	$3k^2 + 6k + 3$

Dimensions of the local finite element spaces $\mathbb{X}_h(T)$ and $\mathbb{T}(T)$.

Implemenation (element-by-element; embedded Trefftz DG)

Let $(\phi_i, \psi_i) \in \mathbb{X}_h(\mathcal{T}_h)$ be basis functions of $\mathbb{X}_h(\mathcal{T}_h)$. Matrix $\mathbf{W}$ (block-diag.) to Stokes operator:

$$(\mathbf{W})_{ij} = (-\Delta \phi_j + \nabla \psi_j, \tilde{\phi}_i)_{\mathcal{T}_h} + (\operatorname{div} \phi_j, \tilde{\psi}_i)_{\mathcal{T}_h},$$

for basis functions $(\tilde{\phi}_i, \tilde{\psi}_i) \in [\mathbb{P}^{k-2}(\mathcal{T}_h)]^d \times \mathbb{P}^{k-1}(\mathcal{T}_h)$.

Compute $\ker(\mathbf{W})$ numerically, e.g. by SVD (altern.: QR)

$$\mathbf{W} = \begin{pmatrix} | & & | & | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_L & \mathbf{u}_{L+1} & \dots & \mathbf{u}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_L & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{v}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_L^T & \text{---} \\ \text{---} & \mathbf{v}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_N^T & \text{---} \end{pmatrix}$$

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Trefftz basis[1] through $\mathbf{T}$:

$$(\phi_i^{\mathbb{T}}, \psi_i^{\mathbb{T}}) = \sum_j \mathbf{T}_{ij}(\phi_j^{\mathbb{T}}, \psi_j) \in \mathbb{T}(\mathcal{T}_h) \subset \mathbb{X}_h(\mathcal{T}_h)$$

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Implemenation (embedded Trefftz DG; particular solutions)

Particular solution: For $\mathbf{F}_i = (f, \tilde{\phi}_i)_{\mathcal{T}_h} + (g, \tilde{\psi}_i)_{\mathcal{T}_h}$ we can compute $(u_{h,p}, p_{h,p})$:

$$\mathbf{W} \cdot \mathbf{u}_p = \mathbf{F} \qquad (\text{e.g. } \mathbf{u}_p = \mathbf{W}^\dagger \mathbf{F}), \qquad (u_{h,p}, p_{h,p}) = \sum_i \mathbf{u}_{p,i}(\phi_i, \psi_i).$$

The role of the pressure in the (homogeneous) Trefftz space

- On $\mathbb{T}$ we have $\nabla p = \nu \Delta u$, i.e. the pressure is determined by the velocity up to a constant (per element). Higher order pressure is determined by velocity.

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- On $\mathbb{T}$ we have $\nabla p = \nu \Delta u$, i.e. the pressure is determined by the velocity up to a constant (per element). Higher order pressure is determined by velocity.
- On $\mathbb{T}$ we have $\operatorname{div} u = 0$ so that we don't need the pressure as Lagrange multiplier for the divergence constraint within each element.
- However, we need the (piecewise constant) pressure as Lagrange multiplier for the mass balance / divergence constraint across the element interfaces.

$$b_h(v_h, p_h) := \underbrace{-(\operatorname{div} v_h, p_h)_{\mathcal{T}_h}}_{=0} + ([\![v_h \cdot n]\!], \{\!\{ p_h \}\!\})_{\mathcal{F}_h}, \quad \text{for } (v_h, q_h) \in \mathbb{T}.$$

Theorem (Inf-Sup stability of $K_h(\cdot,\cdot)$)

For $(v_h, q_h) \in \mathbb{T}$ there holds

$$\sup_{(u_h,p_h)\in\mathbb{T}} \frac{K_h((u_h,p_h),(v_h,q_h))}{\|(u_h,p_h)\|_{\mathbb{T}}} \geq c_{\mathbb{T}} \|(v_h,q_h)\|_{\mathbb{T}} \tag{IS}$$

for constant $c_{\mathbb{T}}$ independent of h, k and ν and hence the Trefftz-DG problems (T-DG) admits a unique solution that depends continuously on the data.[2]

Proof (sketch):

- Saddle point problem with piecew. constant pressures $\mathbb{P}^0$ the Lag. multipliers.
- Inf-sup stability follows from $\mathbb{BDM}^1 \times \mathbb{P}^0 \subset \mathbb{T}$
- (Kernel-)coercivity as usual.

[2] The $\mathbb{T}$-norms are usual DG-type norms.

Lemma (Céa)

Let $(u, p) \in [H^2(\mathcal{T}_h)]^d \times H^1(\mathcal{T}_h)/\mathbb{R}$ be the solution of the Stokes problem (W) and $(u_h, p_h) \in \mathbb{T}_{f,g}$ be the discrete solution to (T-DG). Then, there holds

$$\|(u_h - u, p_h - p)\|_{\mathbb{T}} \lesssim \inf_{(v_h, q_h) \in \mathbb{T}_{f,g}} \|(u - v_h, p - q_h)\|_{\mathbb{T}} \tag{C}$$

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Lemma (Approximation)

Let $(u, p) \in [H^{k+1}(\mathcal{T}_h)]^d \cap [H^1(\Omega)]^d \times H^k(\mathcal{T}_h)/\mathbb{R}$ be the solution of the Stokes problem. There holds

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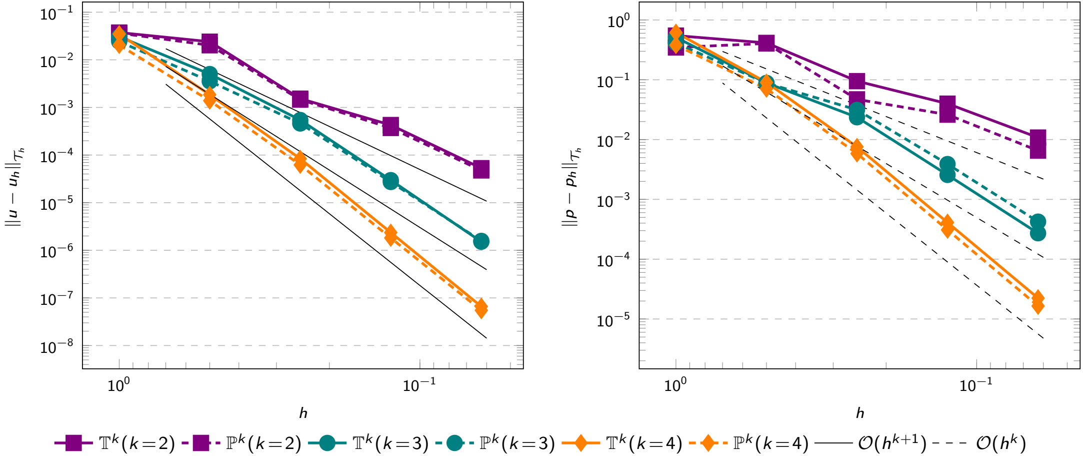
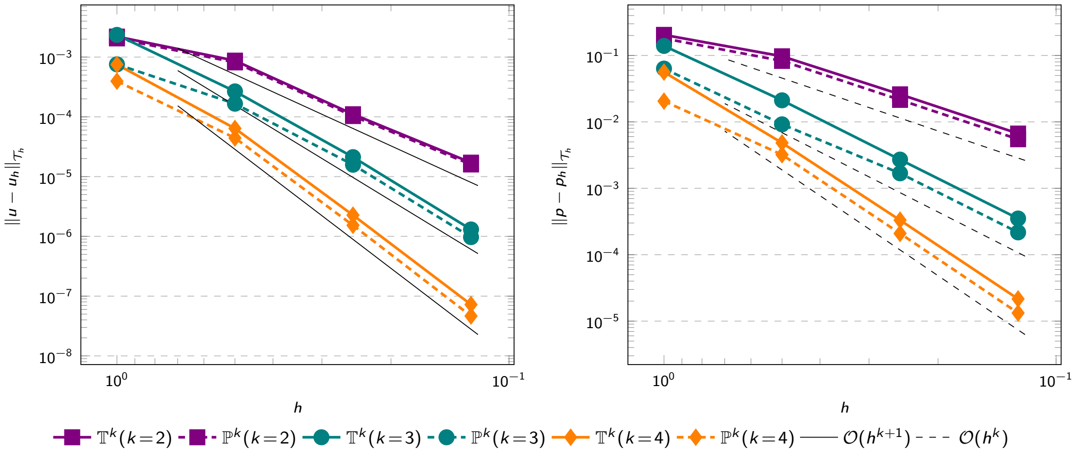
$\rightsquigarrow$ We recover the usual DG convergence rates.

Numerical example (2D)



Numerical example (3D)



Thank you for your attention!

Further details:

Philip L. Lederer, Christoph Lehrenfeld, and Paul Stocker.
Trefftz discontinuous Galerkin discretization for the Stokes problem.
arXiv preprint arxiv:2306.14600, 2023,
accepted for publication in Numerische Mathematik, 2024.

Ongoing work:

- Pressure robust variants
- Variants with (partial) continuity of the velocity
- Extension to Navier-Stokes (non-linear, time-dependent, ...)