The Trefftz approach for unfitted finite element methods

Christoph Lehrenfeld

GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN IN PUBLICA COMMODA SEIT 1737

joint work with Fabian Heimann[1], Erik Schlesinger[2], Paul Stocker[3], Henry v. Wahl[4]

[1]UCL London, [2]University of Göttingen, [3]University of Vienna, [4]University of Jena

May 14, 2024

Contemporary Challenges in Trefftz Methods, from Theory to Applications

Banff International Research Station for Mathematical Innovation and Discovery

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Disclaimer:

Motivation is not primarily wave equations

Instead:

Extending the Trefftz concept to a larger class of applications

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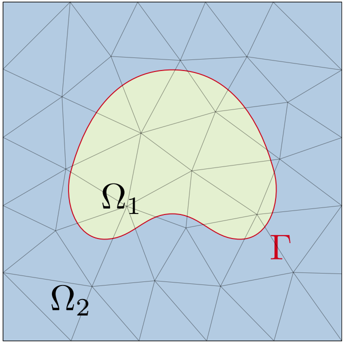
Motivation: Unfitted FEM

Idea of unfitted discretizations:

It can be beneficial to separate geometry and mesh for

- time-dependent geometries (avoiding remeshing)
- avoiding (non-trivial) meshing



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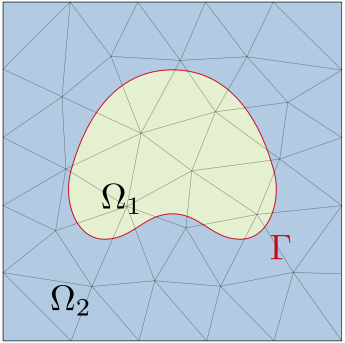
Challenges in unfitted finite elements:

- arbitrary small cuts cause stability issues.
- implementation of unfitted boundary conditions
- geometry description
- cut integration
- time integration
- linear solvers, ...



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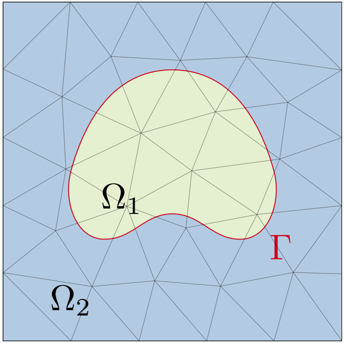

Problem classes:

1️⃣ **one-domain problems** (fictitious domain)

⬜ interface problems

2️⃣ **surface PDEs**

Example problem: Poisson equation on an unfitted mesh

Consider

$$
\begin{aligned}
-\Delta u &= f && \text{in } \Omega, \\
u &= g && \text{on } \;\Gamma = \partial\Omega.
\end{aligned}
$$

as prototypical elliptic problem on an unfitted mesh.

Setting:

- Geometry description independent of the mesh
- $\mathcal{T}_h$: active submesh with cut elements.
- FE space based on $\mathcal{T}_h$

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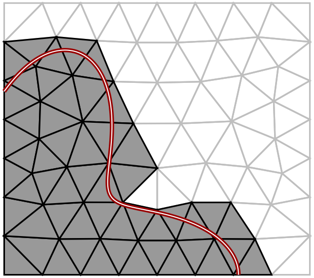

Level set geometry on an unfitted mesh with active elements marked

Setting:

- Geometry description independent of the mesh
- $\mathcal{T}_h$: active submesh with cut elements.
- FE space based on $\mathcal{T}_h$

Starting point:

- discontinuous Galerkin discretisation (Trefftz later)
- weak imposition of boundary conditions through Nitsche

Unfitted DG discretisation (symmetric interior penalty)

Find $u_h \in \mathbb{P}^k(\mathcal{T}_h)$, s.t. $a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in \mathbb{P}^k(\mathcal{T}_h)$.

$$a_h(u,v) = \sum_K \int_{K\cap\Omega} \nabla u \, \nabla v dx + \int_{K\cap\Gamma} \overbrace{-\boldsymbol{n}\cdot\nabla u \; v}^{\text{consistency}} \; \overbrace{-u \; \boldsymbol{n}\cdot\nabla v}^{\text{symmetry}} \; \overbrace{+\lambda k^2 h^{-1} u \; v}^{\text{stability}} ds$$

$$+ \sum_F \int_{F\cap\Omega} -\{\!\{\boldsymbol{n}_F\cdot\nabla u\}\!\}[\![v]\!] - \{\!\{\boldsymbol{n}_F\cdot\nabla v\}\!\}[\![u]\!] + \lambda k^2 h^{-1} [\![u]\!][\![v]\!] ds$$

$$\ell_h(v) = \sum_K \int_{K\cap\Omega} f v dx + \int_{K\cap\Gamma} -g \; \boldsymbol{n}\cdot\nabla v + \lambda k^2 h^{-1} g \; v ds$$

$\{\!\{\cdot\}\!\}$: average across facet, $[\![\cdot]\!]$: jump across facet $\rightsquigarrow$ communication between facets.

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This is IP method on the shape-irregular trimmed mesh $\{K \cap \Omega\}_{K\in\mathcal{T}_h}$

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Find $u_h \in \mathbb{P}^k(\mathcal{T}_h)$, s.t. $a_h(u_h, v_h) + 👻_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in \mathbb{P}^k(\mathcal{T}_h)$.

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Add ghost penalty (👻) stabilization or repair mesh by cell merging.

Unfitted Stabilisation: 👻 Penalty

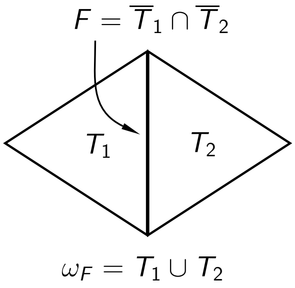
Direct version of the 👻-penalty operator:

$$\text{👻}_h(u,v) = \sum_{F \in \mathcal{F}_h^{\text{👻}}} \frac{\gamma_{\text{👻}}}{h^2} \int_{\omega_F} [\![u]\!]_{\omega_F} [\![v]\!]_{\omega_F} \, dx$$

Here $[\![u]\!]_{\omega_F} := u_1 - u_2$ with $u_i = \mathcal{E} u\big|_{T_i}$,
$\mathcal{E} : \mathbb{P}^m(K) \to \mathbb{P}^m(\mathbb{R}^d)$: canonical pol. ext. to $\mathbb{R}^d$.

This gives us the crucial property

$$\|v\|_{T_1}^2 \lesssim \|v\|_{T_2}^2 + h^2 \, \text{👻}_h^F(v,v).$$



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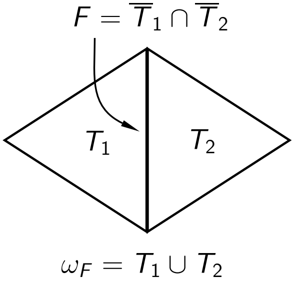

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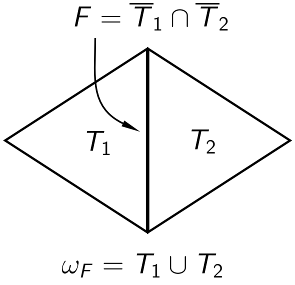

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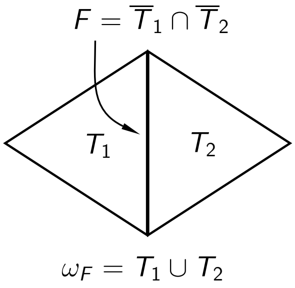

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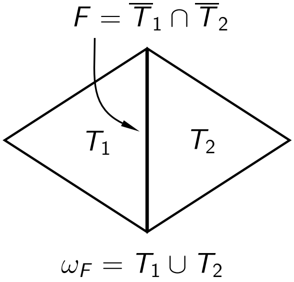

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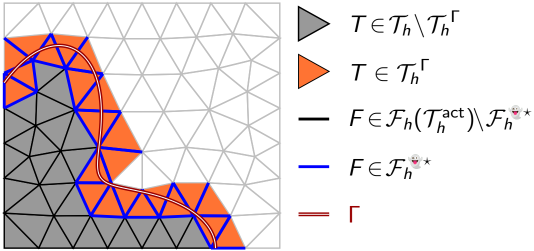
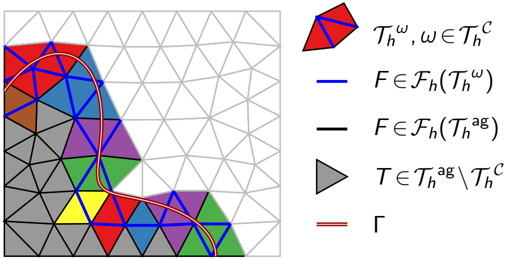
How to choose $\mathcal{F}_h^{\text{👻}}$?

Stabilization facets $\mathcal{F}_h^{👻}$ for the 👻 penalty

Form patches that allow to reach one good element from every (ill-)cut element.



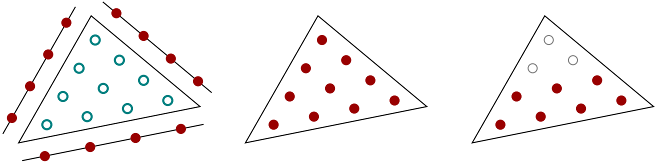


Unfitted (embedded) Trefftz DG

Find $u_h \in \mathbb{T}^k_f(\mathcal{T}_h)$, s.t. $a_h(u_h, v_h) + 👻_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in \mathbb{T}^k_0(\mathcal{T}_h)$ with

$$\mathbb{T}^k_f(\mathcal{T}_h) := \{v_h \in \mathbb{P}^k(\mathcal{T}_h) : -\Delta v_h = \Pi^{k-2} f \text{ on } K \text{ for all } K \in \mathcal{T}_h\}.$$

- Complexity reduction similar to Hybrid DG: $\mathcal{O}(h^{-d}k^d) \rightarrow \mathcal{O}(h^{-d}k^{d-1})$
- Coupling pattern (element-to-element) compatible with 👻 penalty.



Hybrid DG   standard DG   Emb. Trefftz DG

- ●: global dof
- ○: local dof
- ○: removed dof

Recovering the DG error estimates

Lemma (H^1 Estimate)

$u \in H^m(\Omega)$ exact solution with $g \in H^{\frac{1}{2}}(\Gamma)$, $f \in L^2(\Omega)$, $u_h \in \mathbb{T}^k_f(\mathcal{T}_h)$ Trefftz solution. Then,

$$|||u - u_h|||_{\mathcal{A}_h} + |u_h|_{\text{👻}_h} \lesssim h^l \|u\|_{H^{l+1}(\Omega)}, \qquad l = \min\{m-1, k\}.$$

Theorem (L^2 Estimate)

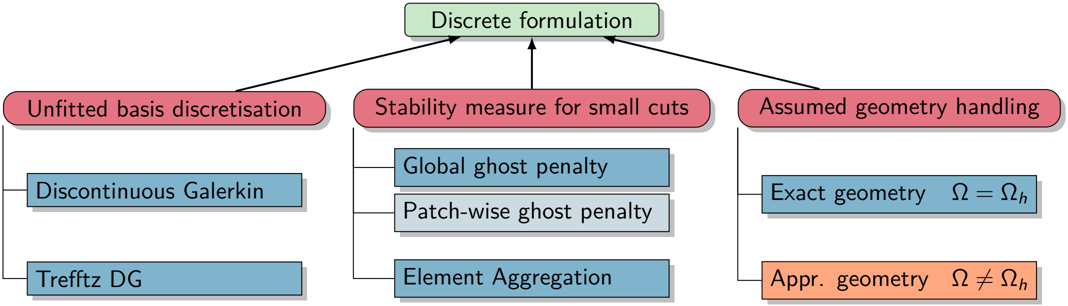
Additionally, assume $L^2(\Omega)$-$H^1(\Omega)$-regularity. Then

$$\|u - u_h\|_\Omega \lesssim h^{l+1} \|u\|_{H^{l+1}(\Omega)}, \qquad l = \min\{m-1, k\}.$$

Proofs follow 'standard' unfitted + 'standard' Trefftz DG methodology:

- Coercivity for sufficiently large $\gamma_{\text{👻}}$ and λ
- (patchwise) averaged Taylor polynomial as interpolant of the extended solution (on $\mathcal{T}_h$)
- Appropriate bound on ghost-penalty contribution.

A few variants



Element aggregation: $\gamma_{👻} \to \infty$ (patch-wise 👻 penalty) $\rightsquigarrow$ patchwise harmonic polynomials.

Embedded Trefftz^2: Extracting harmonic & aggregated polynomials

Compute patchwise kernel of

$$w_h^{\omega}(u,v) = \sum_{K \in \omega} h^2 (\Delta u, \Delta v)_K + \text{👻}_h^{\omega}(u,v)$$

$\ker(w_h^{\omega})$: harmonic functions that are one polynomial on ω.

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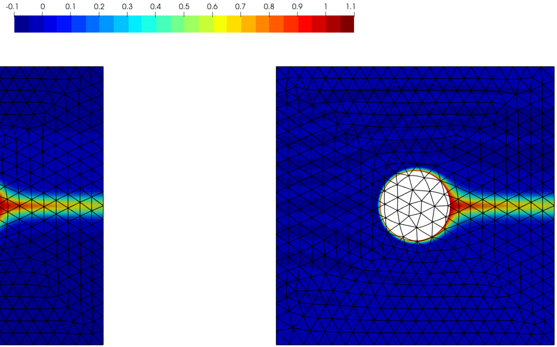
Setup reduces to:

- Setup linear system for unstabilized (w.r.t. cuts) $\mathbb{P}^k(\mathcal{T}_h)$ discretization
- Setup embedded matrix T corresponding to $\ker(w_h^\omega)$
- Setup reduced system ($T^T A T$ and $T^T b$)
- Solve for aggregated Trefftz DG basis
- Reconstruct solution in $\mathbb{P}^k(\mathcal{T}_h)$

Steps can also be patch-localized (to avoid the global DG assembly).

Reference



Right: DG solution; Left: Trefftz DG solution

F. Heimann, C. Lehrenfeld, P. Stocker, H. von Wahl.
Unfitted Trefftz discontinuous Galerkin methods for elliptic boundary value problems.
ESAIM:M2AN 57(5):2803–2833, 2023.

PDEs on surfaces

Example problems:

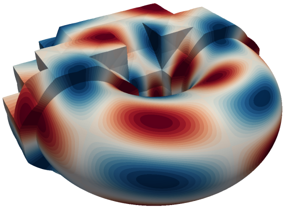
- Laplace-Beltrami equation,

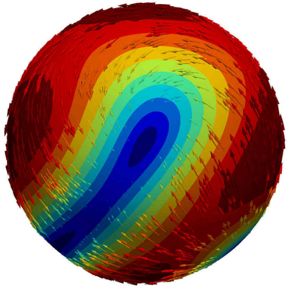
$$u : \Gamma \to \mathbb{R} \qquad -\Delta_\Gamma u = f \quad \text{on } \Gamma$$

- Vector Laplace-Beltrami equation,

$$u : \Gamma \to \mathbb{R}^d \qquad -\Delta_\Gamma u = f, \quad u \cdot \mathbf{n}_\Gamma = 0 \quad \text{on } \Gamma$$

- Surface (Navier-)Stokes equations, ...



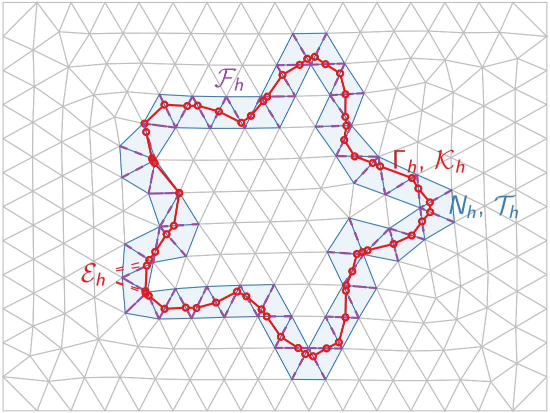


Unfitted ~~FEM~~ DG for Surface PDEs, the Trace ~~FEM~~ DG

Unfitted mesh entities:

- active mesh: $\mathcal{T}_h$ (domain: N_h)
- (background) FE space: $\mathbb{P}^k(\mathcal{T}_h)$
- irregular! surface mesh: $\mathcal{K}_h$ (surface: Γ_h)
- edges[a] of surface mesh: $\mathcal{E}_h$
- (active) facets of background mesh: $\mathcal{F}_h$

[a] in the 3D case, vertices otherwise



TraceFEM on irregular cut surface mesh for $-\Delta_\Gamma u = f$ on Γ

Naive discrete bilinear form for (negative) Laplace-Beltrami operator on $\mathbb{P}^k_{\text{cont}}(\mathcal{T}_h)$:

$$a_h(u,v) = \sum_{K \in \mathcal{K}_h} \int_K \nabla_{\Gamma_h} u \cdot \nabla_{\Gamma_h} v, \qquad a_h(u,u) = ||\nabla_{\Gamma_h} u||^2_{\Gamma_h}$$

- $||\nabla_{\Gamma_h} \cdot ||_{\Gamma_h}$ has a large (near-) kernel: vol. fcts. vanishing on surface

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- $\|\nabla_{\Gamma_h} \cdot \|_{\Gamma_h}$ has a large (near-) kernel: vol. fcts. vanishing on surface

Remedy: Normal gradient volume stabilisation

- PDE acts in tangential direction of the surface. Add equation in normal direction:

$$j_h(u,v) = \sum_{T \in \mathcal{T}_h} \int_T \lambda^n (\nabla u \cdot \mathbf{n}_\Gamma)(\nabla v \cdot \mathbf{n}_\Gamma)$$

$\mathbf{n}_\Gamma$: quasi-normal vector (extension of surface normal to neighborhood).

Interior penalty Trace DG on irregular cut surface mesh

$$\underbrace{\sum_{K\in\mathcal{K}_h}\int_K \nabla_{\Gamma_h} u \cdot \nabla_{\Gamma_h} v}_{=a_h(u,v)} + \underbrace{\sum_{T\in\mathcal{T}_h}\int_T \lambda^n (\nabla u\cdot \mathbf{n}_\Gamma)(\nabla v \cdot \mathbf{n}_\Gamma)}_{=j_h(u,v)}$$

For $\lambda^n \gtrsim h$ there holds

$$\|u_h\|_\Gamma^2 + j_h(u_h, u_h) \gtrsim h^{-1}\|u_h\|_{N_h}^2, \qquad u_h \in \mathbb{P}^k_{\text{cont}}(\mathcal{T}_h)$$

$\rightsquigarrow$ For $h^{-1} \gtrsim \lambda^n$: optimal error and condition number bounds independent of cut position.

J. Grande, C. Lehrenfeld, A. Reusken.
Analysis of a high-order trace finite element method for PDEs on level set surfaces.
SINUM 56(1):228–255, 2018.

Interior penalty Trace DG on irregular cut surface mesh

$$a_h(u,v) \rightsquigarrow \sum_{K\in\mathcal{K}_h}\int_K \nabla_{\Gamma_h} u \cdot \nabla_{\Gamma_h} v - \sum_{E\in\mathcal{E}_h}\int_E \{\!\{ n_E \cdot \nabla u \}\!\} [\![v]\!] - \sum_{E\in\mathcal{E}_h}\int_E \{\!\{ n_E \cdot \nabla v \}\!\} [\![u]\!] + \frac{\lambda}{h}\sum_{E\in\mathcal{E}_h}\int_E [\![u]\!][\![v]\!]$$

Stability issues:

- λ scales with shape regularity that may become unbounded.

[1] not all volume d.o.f.s needed this time

Interior penalty Trace DG on irregular cut surface mesh

$$a_h(u,v) \rightsquigarrow \sum_{K\in\mathcal{K}_h}\int_K \nabla_{\Gamma_h} u \cdot \nabla_{\Gamma_h} v - \sum_{E\in\mathcal{E}_h}\int_E \{\!\{ n_E \cdot \nabla u \}\!\} [\![v]\!] - \sum_{E\in\mathcal{E}_h}\int_E \{\!\{ n_E \cdot \nabla v \}\!\} [\![u]\!] + \frac{\lambda}{h}\sum_{E\in\mathcal{E}_h}\int_E [\![u]\!][\![v]\!]$$

Stability issues:

- λ scales with shape regularity that may become unbounded.

Remedy: Add stabilisations:

- Normal gradient volume stabilisation (as before):

$$j_h(u,v) = \sum_{T\in\mathcal{T}_h}\int_T \lambda^n (\nabla u \cdot \mathbf{n}_\Gamma)(\nabla v \cdot \mathbf{n}_\Gamma),$$

- Borrow stability from neighbouring volume elements[1].

$$\text{👻}_h(u,v) = \sum_{F\in\mathcal{F}_h}\int_F \frac{\lambda_0^{\text{👻}}}{h^2}[\![u]\!][\![v]\!] + \int_F \lambda_1^{\text{👻}} [\![\mathbf{n}_F \cdot \nabla u]\!][\![\mathbf{n}_F \cdot \nabla v]\!]$$

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Trace DG analysis / Trefftz Trace DG

For $\lambda^n \simeq h^{-1}$, $\lambda_0^{\text{👻}}, \lambda_1^{\text{👻}} \gtrsim 1$ optimal error and condition number bounds indep. of cut position.

$$\lambda^n \to \infty \rightsquigarrow u_h \in \ker(j_h) = \{v \in \mathbb{P}^k(\mathcal{T}_h) \mid \nabla v \cdot \mathbf{n}_\Gamma = 0\} \leftarrow \text{ Trefftz DG space}$$

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Analysis of TraceFEM reveals:

- It suffices to use $\mathbf{n}_h = \mathbf{n}_\Gamma + \mathcal{O}(h)$ in $j_h(\cdot,\cdot)$ for stability.
- It also suffices to penalize $\Pi^Q \mathbf{n}_h \cdot \nabla u$ in $j_h(\cdot,\cdot)$ for stability with $Q = \mathbb{P}^{k-1}(\mathcal{T}_h)$.

Modified Trace DG $\rightsquigarrow$ Trace Trefftz DG

Relaxed normal gradient stabilization:

$$j_h^*(\lambda^n; u, v) = \sum_{T \in \mathcal{T}_h} \int_T \lambda^n (\Pi^Q \nabla u \cdot \mathbf{n}_\Gamma)(\Pi^Q \nabla v \cdot \mathbf{n}_\Gamma), \qquad \rightsquigarrow \ker(j_h) = \{v \in \mathbb{P}^k(\mathcal{T}_h) \mid \Pi^Q \nabla v \cdot \mathbf{n}_\Gamma = 0\}$$

Then, the following three formulations are equivalent:

1️⃣

$u_h^1 := \lim_{\lambda^n \to \infty} u_\lambda^1$ with
Find $u_\lambda^1 \in \mathbb{P}^k(\mathcal{T}_h)$ s.t.

$$a_h(u_\lambda^1, v_h) + j_h^*(\lambda^n; u_\lambda^1, v_h) = f_h(v_h)$$

for all $v_h \in \mathbb{P}^k(\mathcal{T}_h)$.

2️⃣

Find $u_h^2, p_h^2 \in \mathbb{P}^k(\mathcal{T}_h) \times \mathbb{P}^{k-1}(\mathcal{T}_h)$ s.t.

$$a_h(u_h^2, v_h) + b_h(v_h, p_h^2) = f_h(v_h),$$
$$b_h(u_h^2, q_h) = 0,$$

for all $v_h, q_h \in \mathbb{P}^k(\mathcal{T}_h) \times \mathbb{P}^{k-1}(\mathcal{T}_h)$,
$b_h(u, q) = \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \mathbf{n}_\Gamma \cdot q$.

3️⃣

Find $u_h^3 \in \ker(j_h^*)$ s.t.

$$a_h(u_h^3, v_h) = f_h(v_h)$$

for all $v_h \in \ker(j_h^*)$.

Analysis, Dimension reduction, implementation

Analysis of 1️⃣ and equivalence of 1️⃣, 2️⃣, 3️⃣ yields quasi best approximation results:

$$\|u - u_h\|_{1,\Gamma,h} \lesssim \inf_{v_h \in \ker(j_h^*)} \|u - v_h\|_{1,\Gamma,h}$$

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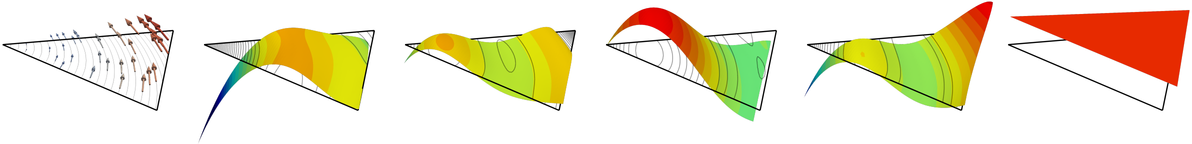

E. Schlesinger,
Embedded Trefftz Trace DG Methods for PDEs on unfitted Surfaces.
Master's thesis, University of Göttingen, 2023.

Summary & Outlook

Unfitted Trefftz DG for elliptic PDEs

- penalty stabilisation harmonizes well with Trefftz DG (but not with Hybrid DG)
- Embedded Trefftz DG and aggregated FEM/DG are similar in virtue
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- 🤔 Vector-Laplace/Stokes: $-\mathcal{L}_\Gamma v = 0$, $\mathbf{n}_\Gamma \cdot \nabla v = 0$ and $v \cdot \mathbf{n}_\Gamma = 0$ in one Trefftz space?
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Thank you for your attention!