Tuning the Role of Unknowns in Higher Order (Conforming- and Non-conforming) Finite Element Methods

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SEIT 1737

joint w. Philip L. Lederer[1], Paul Stocker[2], Igor Voulis[3] and others

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Outline

Outline

- Example Application Fields for Higher Order Finite Element Methods
- The Roles of Unknowns in Conforming FEM: A Local-Global Splitting
- Unknowns in Non-conforming FEM (DG)
- Hybridization and a Local-Global Splitting for DG methods
- Trefftz-like DG Methods
  - Classical Trefftz DG Methods
  - Generalizations: Embeddings and a Local-Global Splitting
- Conclusion & Outlook

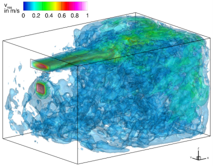
Incompressible flows

$$\begin{cases} \frac{\partial}{\partial t} u + \operatorname{div}(-\nu \nabla u + u \otimes u) + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega. \end{cases}$$

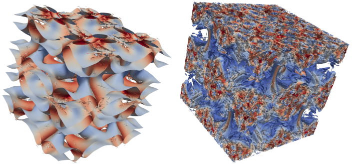
+ initial / boundary cond.

Num. challenges:

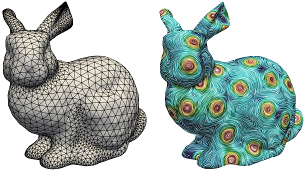
- Structure-preservation
- Convection domination
- ...



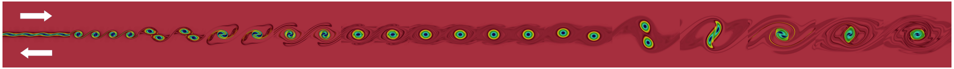
indoor simulation[1]



turbulence[2]



flows on surfaces[3]



Kelvin-Helmholtz instability[4]

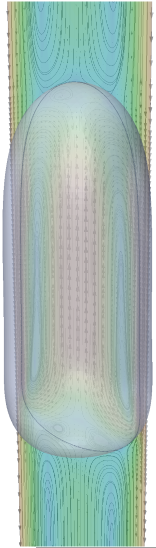
[1] with R. Gritzki, M. Rösler & C. Felsmann, TU Dresden

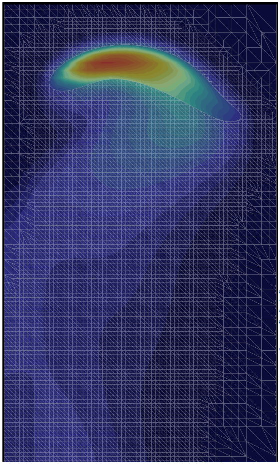
[2] with N. Fehn, M. Kronbichler (TU München), G. Lube & P.W. Schroeder (Uni Göttingen)

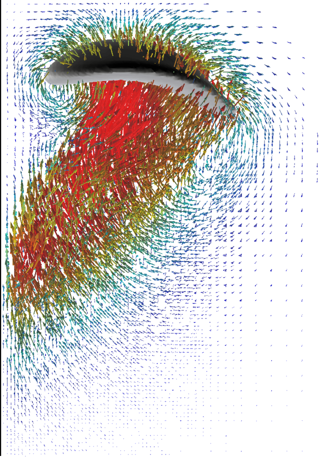
[3] with P.L. Lederer & J. Schöberl (TU Wien) – & T. Brüers & M. Wardetzky

[4] with V. John (WIAS Berlin), P.L. Lederer, J. Schöberl (TU Wien), G. Lube, P.W. Schroeder (U. Gö.)

Moving domain problems[5]

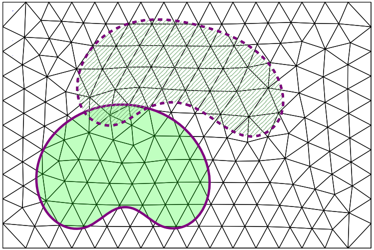






Num. challenges:

- geometry tracking/capturing
- time integration
- flexibility & robustness w.r.t. geom. changes
- ...

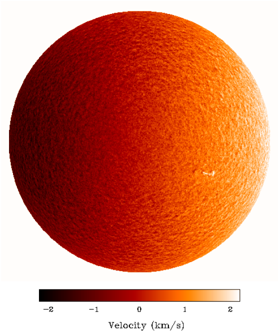


$$\dot{u} - \mathcal{L}u = f \text{ in } \Omega(t)$$

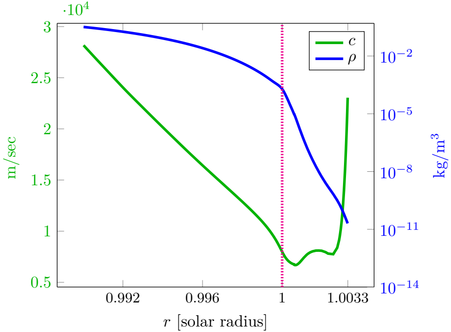
[5] with Reusken (Aachen), Olshaniskii (Houston), Massing (Trondheim), Preuß (UCL), v.Wahl (Jena)

Computational Helioseismology[6]

$$\rho(-i\omega + \mathbf{u}\cdot\nabla + \Omega\times)^2\boldsymbol{\xi} - (\nabla + \mathbf{q})(\rho c^2(\operatorname{div} + \mathbf{q}\cdot)\boldsymbol{\xi}) - i\gamma\rho\omega\boldsymbol{\xi} + ..(compact).. = \mathbf{s} \quad \text{in} \quad \Omega.$$



dopplergram



ρ: density & c: sound speed

Num. challenges:

- indefinite system
- high resolution demands
- highly varying coefficients
- domain truncation (reflections)
- ...

[6] with T. Hohage, M. Halla & T. van Beeck (Uni Göttingen) D. Fournier & L. Gizon (MPS Göttingen)

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CRC 1456 MATHEMATICS OF EXPERIMENT

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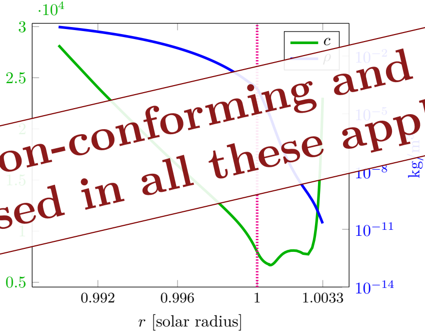

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Higher order non-conforming and conforming FEM are used in all these applications.

[6] with T. Hohage, M. Halla & T. van Beeck (Uni Göttingen) D. Fournier & L. Gizon (MPS Göttingen)

Outline

Finite Element Discretizations (with some simplifications)

Generic linear PDE problem

$$\mathcal{L}u = \ell \quad \text{in } \Omega \qquad + \text{b.c.}$$

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Continuous $\rightsquigarrow$ discrete variational formulation

$$\begin{array}{llll}
\text{Find } u \in V, \text{ s.t. } a(u,v) = \ell(v) & \forall\, v \in W, & & V,\ W \text{ Hilbert spaces.} \\
\quad\ \ \downarrow \qquad\quad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow & & & \\
\text{Find } u_h \in V_h, \text{ s.t. } a_h(u_h,v_h) = \ell_h(v_h) & \forall\, v_h \in V_h, & & V_h \text{ finite dimensional Hilbert space.}
\end{array}$$

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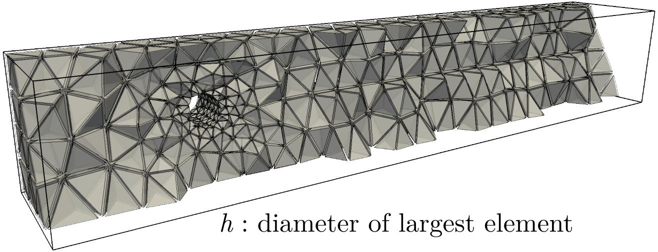
Design choices in a FE discretization

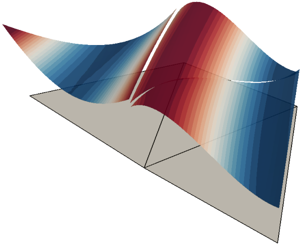
- How to choose V_h?
- How to choose $a_h(\cdot,\cdot)$, $f_h(\cdot)$?
- solvers, preconditioners, implementation, ...

Domain decomposition into a mesh of simple elements



h : diameter of largest element

Approximation



- With element-wise polynomials of degree k we ideally obtain

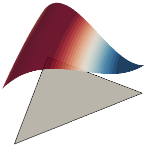
$$\inf_{v_h \in V_h} \|u - v_h\|_h \lesssim \left(\frac{h}{k}\right)^k \|u\|_{H^{k+1}(\Omega)}$$

- Convergence in h and/or k possible.
- If regularity permits, increasing k should be prefered over decreasing h.

The construction of finite elements: The conforming way ($V_h \subset V$)

- Pick a polynomial space on each element

Polynomial on one element



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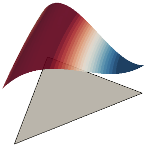
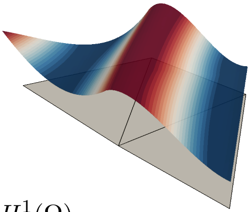


Here: continuity ($V = H^1(\Omega)$)



$$V_h = \{v \in C(\Omega) \mid u|_T \in \mathcal{P}^k(T)\} \subset V = H^1(\Omega)$$

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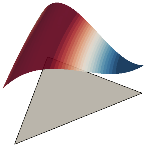

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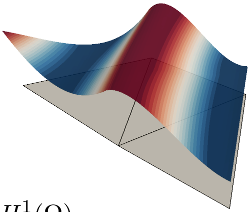

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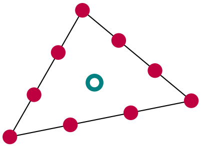
Realization through splitting of `dofs` / basis functions

- interior basis fcts. (bubbles) vanish on skeleton (local `dofs`)
- shared `dofs` are shared on element interfaces (skeleton)



: local `dof`

: shared `dof`

Higher Order Methods

- The approximation error per `dof` is smaller
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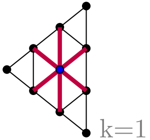
Simple illustration

Low order / small k

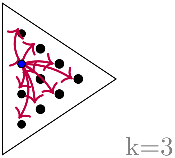
$\mathcal{O}(1)$ `dofs` per geom. entity



$\mathcal{O}(1)$ couplings per `dof`

High order / large k

$\binom{k+d}{d} \sim \mathcal{O}(k^d)$ `dofs` per geom. entity



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Different Roles of Unknowns in Continuous Scalar FEM

Unknowns scale with $\mathcal{O}(k^d)$ (w.r.t. k), but

- element local `dofs` scale with $\mathcal{O}(k^d)$, but only couple locally
- only $\mathcal{O}(k^{d-1})$ `dofs` on the skeleton

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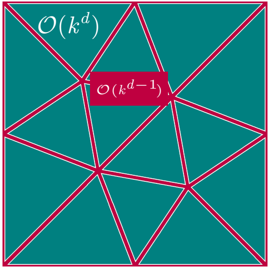
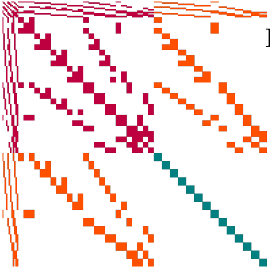


triangular mesh



sparsity pattern for $k = 5$

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Decompose FE space into

- globally coupled/skeleton parts and
- local parts:

$V_h = \mathbb{T}_h \oplus \mathbb{L}_h.$

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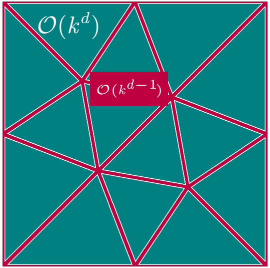

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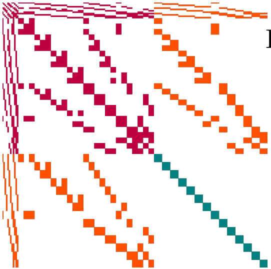

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$$\begin{pmatrix} A_{\mathbb{TT}} & A_{\mathbb{TL}} \\ A_{\mathbb{LT}} & A_{\mathbb{LL}} \end{pmatrix} \cdot \begin{pmatrix} u_{\mathbb{T}} \\ u_{\mathbb{L}} \end{pmatrix} = \begin{pmatrix} f_{\mathbb{T}} \\ f_{\mathbb{L}} \end{pmatrix}$$

$A_{\mathbb{TT}}$: $\mathcal{O}(k^{2d-2})$ entries
$A_{\mathbb{LT}}, A_{\mathbb{TL}}$: $\mathcal{O}(k^{2d-1})$ entries
$A_{\mathbb{LL}}$: $\mathcal{O}(k^{2d})$ entries, block diagonal

Static condensation for conforming FEM

Simple calculation yields ($A_{\mathbb{LL}}^{-1}$ is cheap to form)

$$\begin{pmatrix} A_{\mathbb{TT}} & A_{\mathbb{TL}} \\ A_{\mathbb{LT}} & A_{\mathbb{LL}} \end{pmatrix} = \begin{pmatrix} I & A_{\mathbb{TL}} A_{\mathbb{LL}}^{-1} \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} S & 0 \\ 0 & A_{\mathbb{LL}} \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ A_{\mathbb{LL}}^{-1} A_{\mathbb{LT}} & I \end{pmatrix} \text{ with } S = A_{\mathbb{TT}} - A_{\mathbb{TL}} A_{\mathbb{LL}}^{-1} A_{\mathbb{LT}}$$

Costs for solving linear systems:

- Global solution with S ($\mathcal{O}(k^{d-1} \cdot \# \mathcal{T}_h)$ `dofs`) (and setup)
- Setup and application of ${A_{\mathbb{LL}}}^{-1}$ ($\# \mathcal{T}_h$ times $\mathcal{O}(k^d)$ `dofs`; parallelizable)

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Summary:

Static condensation allows to exploit a local-global `dof` splitting
$\rightsquigarrow$ increases efficiency 🚀 of higher order conforming FEM

Outline

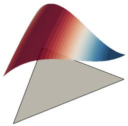
Non-conforming (Discontinuous Galerkin) FEM

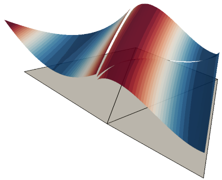
$$\mathcal{L}u = \ell \text{ in } \Omega \subset \mathbb{R}^d + \text{boundary conditions.}$$

A typical standard DG discretization ($V_h = \mathbb{P}^k(\mathcal{T}_h) \not\subset V$):

$$\text{Find } u_h \in \mathbb{P}^k(\mathcal{T}_h), \quad \text{s.t. } a_h(u_h, v_h) = \ell_h(v_h) \qquad \forall v_h \in \mathbb{P}^k(\mathcal{T}_h)$$

with disconnected polynomial spaces $\mathbb{P}^k(K)$. Regularity is imposed weakly through $a_h(\cdot, \cdot)$.





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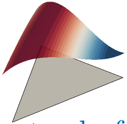
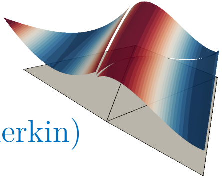




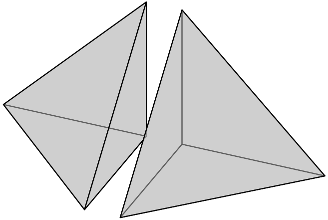
Motivation for DG (instead of continuous Galerkin)

Exploiting flexibility for ...

- ... conservation properties (test function χ_K)
- ... simple stability mechanism for non-symm./non-lin. problems (e.g. convection)
- ... simplicity of data structures / space construction (polygonal meshes)
- ... local bases, block diagonal mass matrices, ...

Example: Standard DG for Poisson (Symm. int. pen. DG)

$$\mathcal{L}u = -\Delta u = \ell \text{ in } \Omega, \quad g = u \text{ on } \partial\Omega.$$



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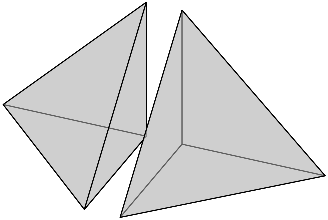


DG discretization

Find $u_h \in \mathbb{P}^k(\mathcal{T}_h)$, s.t. $a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in \mathbb{P}^k(\mathcal{T}_h)$ with

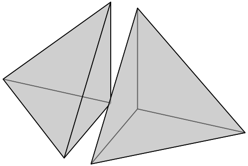
$$a_h(u,v) = \sum_K \int_K \nabla u \nabla v \, dx + \sum_{F \in \mathcal{F}_h^{\text{int}}} \int_F \overbrace{-\{\!\{\partial_{\mathbf{n}} u\}\!\}[\![v]\!]}^{\text{consistency}} \overbrace{-\{\!\{\partial_{\mathbf{n}} v\}\!\}[\![u]\!]}^{\text{symmetry}} \overbrace{+ \alpha p^2 h^{-1} [\![u]\!][\![v]\!]}^{\text{stability}} \, ds$$

$$+ \sum_{F \in \mathcal{F}_h^{\text{bnd}}} \int_F -\partial_{\mathbf{n}} u \; v - \partial_{\mathbf{n}} v \; u + \alpha p^2 h^{-1} uv \, ds$$

$$\ell_h(v) = \sum_K \int_K fv \, dx + \sum_{F \in \mathcal{F}_h^{\text{bnd}}} \int_F (-\partial_{\mathbf{n}} v + \alpha p^2 h^{-1} v) g \, ds.$$

Communication between neighbors with average ($\{\!\{\cdot\}\!\}$) and jump ($[\![\cdot]\!]$) across facets.

Solving linear systems with DG



Issues of DG methods (compared to CG)

- ⚠️ Breaking up continuity ⇝ more unknowns (`dofs`)
- ⚠️ Essentially all element `dofs` couple with all neighbor `dofs`
  ⇝ even more couplings, i.e. more non-zero entries (`nzes`)
- 😢 No local-global splitting ⇝ no static condensation

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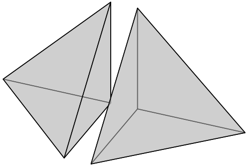

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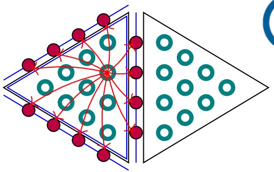
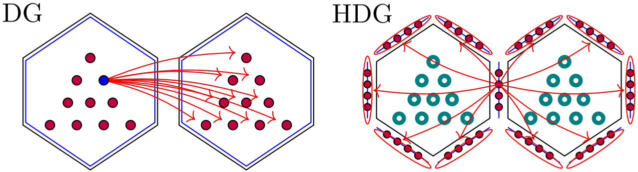
Remedies to re-introduce a local-global splitting?

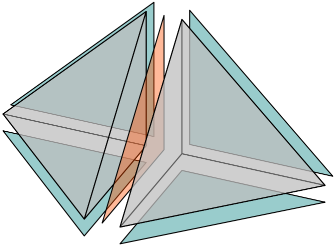
Outline

Hybridization of DG methods: The concept



- add discontinuous facet (skeleton) unknowns λ_h
- avoid direct communication between elements ($\mathcal{O}(k^d) \not\leftrightsquigarrow \mathcal{O}(k^d)$)
- communication between element `dofs` and facet `dofs` instead ($\mathcal{O}(k^d) \leftrightsquigarrow \mathcal{O}(k^{d-1})$)
- apply static condensation





[7] B. Cockburn, J. Gopalakrishnan, R. Lazarov, *Unified hybridization of [dG], [...] for [2nd] order elliptic problems.* SINUM, 2009

Hybrid DG in primal formulation (Hybrid Symm. int. pen. DG)

$$\mathcal{L}u = -\Delta u = \ell \text{ in } \Omega, \quad g = u \text{ on } \partial\Omega.$$

HDG discretization

Find $\underline{u}_h = (u_h, \lambda_h) \in \mathbb{P}^k(\mathcal{T}_h) \times F^k_{h,D}(\mathcal{F}_h)$, s.t. $a_h(\underline{u}_h, \underline{v}_h) = \ell_h(\underline{v}_h) \quad \forall \underline{v}_h = (v_h, \mu_h) \in \mathbb{P}^k(\mathcal{T}_h) \times F^k_{h,0}(\mathcal{F}_h)$

with $F^k_h(\mathcal{F}_h) = \{v \in L^2(\mathcal{F}_h) \mid v|_F \in \mathbb{P}^k(F) \ \forall F \in \mathcal{F}_h\}$, and

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$$\ell_h(\underline{v}_h) = \sum_{K\in\mathcal{T}_h} \int_K f v_h \, dx$$

$[\![\underline{u}_h]\!] = u_h - \lambda_h$: jump between el. trace & facet function.

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no "DG jump" and no average across facets $\rightsquigarrow$ communication stays local.

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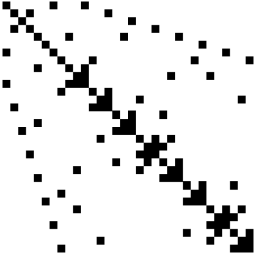
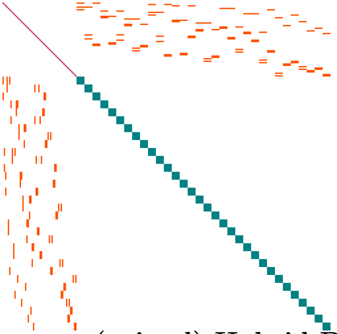
Hybrid DG: static condensation for DG



(primal) DG
2D, $k = 8$



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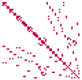

HDG
(cond.)

💡 Hybridization allows to re-introduce static condensation in DG formulations.

❗ Dominating costs depend on skeleton `dofs`: $\mathcal{O}(k^{d-1})$. – further tuning possible.

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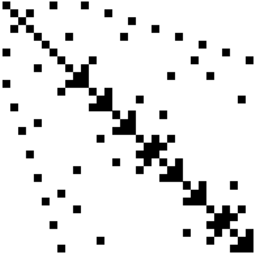

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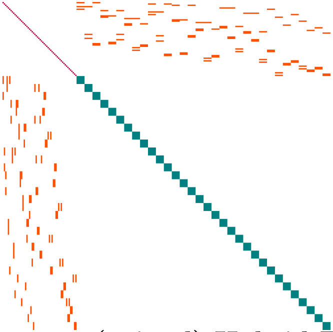

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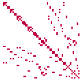

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- 💡 Hybridization allows to re-introduce static condensation in DG formulations.
- ❗ Dominating costs depend on skeleton `dofs`: $\mathcal{O}(k^{d-1})$. – further tuning possible.
- 💪 Applicable to most DG discretizations ⇝ key for efficiency of $H(\text{div})$ flow solvers[8].

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- Example Application Fields for Higher Order Finite Element Methods
- The Roles of Unknowns in Conforming FEM: A Local-Global Splitting
- Unknowns in Non-conforming FEM (DG)
- Hybridization and a Local-Global Splitting for DG methods
- Trefftz-like DG Methods
  - Classical Trefftz DG Methods
  - Generalizations: Embeddings and a Local-Global Splitting
- Conclusion & Outlook

Reduction of global `dofs` using ~~Hybrid~~ Trefftz DG

Alternative to Hybridization

So far: reduce globally coupled `dofs` by static condensation

🎯 Goal of classical Trefftz DG: reduce all `dofs`, s.t.

- approximation (order) is preserved
- (all) `ndofs`: $\mathcal{O}(k^d) \rightsquigarrow \mathcal{O}(k^{d-1})$

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🤔 How?

💡 Trefftz DG idea:

- Replace full polynomial spaces
- Use element-local PDE solutions
- Exploit flexibility of DG (no continuity `dofs` required)

Example: Trefftz DG for Laplace[9]

$$\mathcal{L}u = -\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

DG discretization

Find $u_h \in \mathbb{P}^k(\mathcal{T}_h)$, s.t. $a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in \mathbb{P}^k(\mathcal{T}_h)$ with

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Trefftz DG discretization

$$\mathbb{T}^k(\mathcal{T}_h) := \ker(-\Delta) = \{v \in \mathbb{P}^k, \ \mathcal{L}v = 0 \text{ (pointwise) on each } K \in \mathcal{T}_h\} \subset \mathbb{P}^k(\mathcal{T}_h).$$

Find $u_{\mathbb{T}} \in \mathbb{T}^k(\mathcal{T}_h)$, s.t. $a_h(u_{\mathbb{T}}, v_{\mathbb{T}}) = \ell_h(v_{\mathbb{T}}) \quad \forall v_{\mathbb{T}} \in \mathbb{T}^k(\mathcal{T}_h)$.

---

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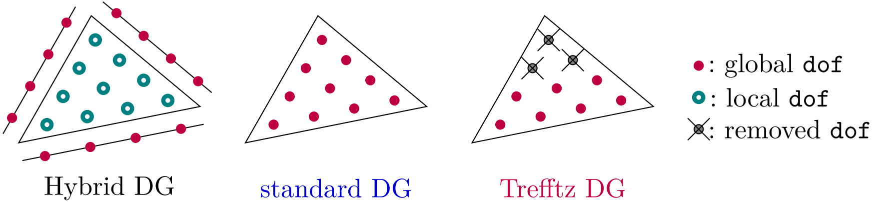
Reduction of computational costs (Laplace)

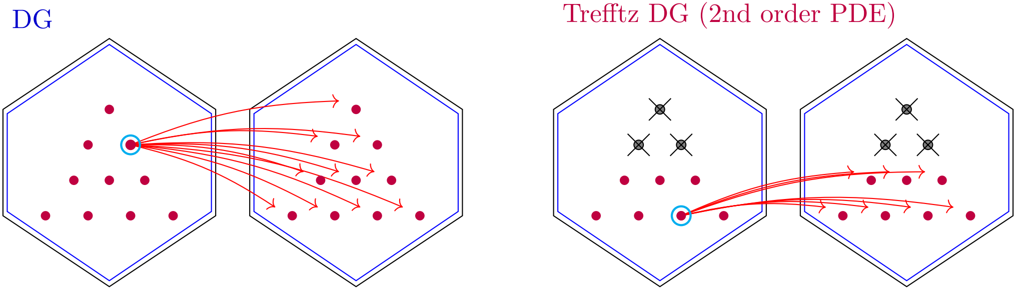
Counting of `ndofs` (triangular mesh, $\mathcal{L} = -\Delta$)

- $N = \dim(\mathbb{P}^k) = \#\mathcal{T}_h \cdot \frac{(k+1)(k+2)}{2} \sim \mathcal{O}(k^d)$,
- $L = \dim(\text{range}\,(\mathcal{L})) = \mathbb{P}^{k-2}(\mathcal{T}_h) = \#\mathcal{T}_h \cdot \frac{(k-1)k}{2} \sim \mathcal{O}(k^d)$,
- $M = \dim(\mathbb{T}^k(\mathcal{T}_h)) = \dim(\ker\,(\mathcal{L})) = N - L = \#\mathcal{T}_h \cdot (2k+1) \sim \mathcal{O}(k^{d-1})$



Trefftz DG achieves reduction $\mathcal{O}(k^d) \rightsquigarrow \mathcal{O}(k^{d-1})$ (for all `dofs`)!

Trefftz DG coupling pattern



Difficulties with classical Trefftz DG methods

Disadvantages and limitations

- 🔴 New basis for each diff operator $\mathcal{L}$ / PDE
- ⚪ Conditioning often problematic
- ❌ Not (directly) suitable for inhomogeneous equations $f \neq 0$
- ❌ Not (directly) suitable for non-constant coefficients, e.g. $\mathcal{L} = \operatorname{div}(\alpha \nabla \cdot)$

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So far: Method not flexible, used only in special cases

❓ 🤔 Can we turn Trefftz into a general purpose tool 🛠️ ❓

Outline

Construction of a Trefftz Embedding in DG space

Avoid setting up Trefftz basis from scratch! $\mathbb{T}^k(\mathcal{T}_h) \subset \mathbb{P}^k$, $M = \dim(\mathbb{T}^k) < \dim(\mathbb{P}^k) = N$

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Let $\mathcal{L} : \mathbb{P}^k(\mathcal{T}_h) \to Q_h'$ with $Q_h = \bigtimes_{K\in\mathcal{T}_h} Q_K = \text{span}\{\varphi_i\}_{i=1,..,L}$. Then with

$$(\mathbf{W})_{ij} = w_h(\phi_j, \varphi_i) = \sum_{K\in\mathcal{T}_h} \langle \mathcal{L}\phi_j, \varphi_i|_K \rangle, \quad i = 1, \dots, L, \quad j = 1, \dots, N \Longrightarrow \mathbf{T}\cdot\mathbb{R}^M = \ker(\mathbf{W}).$$

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Compute $\ker(\mathbf{W})$ numerically by SVD (or QR):

- $$\mathbf{W} = \begin{pmatrix} | & & | & | & & | \\ \mathbf{w}_1 & \dots & \mathbf{w}_L & \mathbf{w}_{L+1} & \dots & \mathbf{w}_N \\ | & & | & | & & | \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_L & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{---} & \mathbf{v}_1^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_L^T & \text{---} \\ \text{---} & \mathbf{v}_{L+1}^T & \text{---} \\ & \vdots & \\ \text{---} & \mathbf{v}_N^T & \text{---} \end{pmatrix}$$

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- Computations element-by-element (and in parallel), $\mathbf{W}$ block-diagonal, orthogonal $\mathbf{T}$-columns

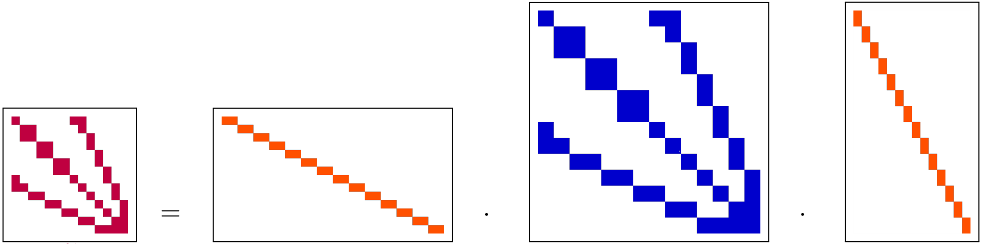
[10] C.L., P. Stocker. *Embedded Trefftz Discontinuous Galerkin methods.* IJNME, 2023.

Setup of Embedded Trefftz DG linear systems

Standard DG setting matrix/vector

$$(\mathbf{A})_{ij} = a_h(\phi_j, \phi_i) \;\; i,j = 1, \ldots, N, \qquad (\boldsymbol{\ell}_h)_i = \ell_h(\phi_i) \;\; i = 1, \ldots, N$$

Embedded Trefftz DG linear algebra (example with $k = 5$)



$$\tilde{\mathbf{A}} = \mathbf{T}^T \cdot \mathbf{A} \cdot \mathbf{T}$$

$$\tilde{\mathbf{A}}\, \mathbf{w}_{\mathbb{T}} = \mathbf{T}^T \mathbf{A} \mathbf{T}\, \mathbf{w}_{\mathbb{T}} = \mathbf{T}^T \boldsymbol{\ell}_h. \qquad \text{(assembly still element-by-element / facet-by-facet)}$$

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[10] C.L., P. Stocker. *Embedded Trefftz Discontinuous Galerkin methods.* IJNME, 2023.

Benefits of Embedded Trefftz DG

So far:

Embedded Trefftz DG is (merely) an implementation trick for existing polynomial Trefftz.

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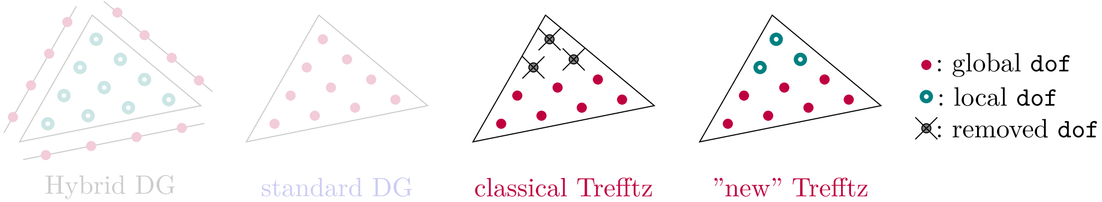
Next:

Claim: Embedded Trefftz DG ...

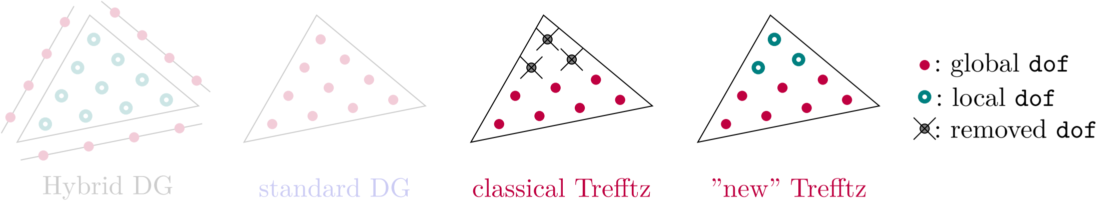
1. ... inherites conditioning properties from DG scheme ($\kappa_2(\tilde{\mathbf{A}}) = \kappa_2(\mathbf{T}^T\mathbf{A}\mathbf{T}) \leq \kappa_2(\mathbf{A})$.)
2. ... allows to treat inhomogeneous PDEs
3. ... allows to conveniently implement weak Trefftz spaces
   $\rightsquigarrow$ treat PDEs where no (suitable) polynomial Trefftz spaces exists

Re-introduce local dofs to Trefftz DG



[11] A. Lozinski. *A primal [dG] method with static condensation on very general meshes.*, Numerische Mathematik, 2019
[12] C.L., P. Stocker, I. Voulis. *A unified framework for Trefftz-like discretization methods*, in preparation, 2024

Re-introduce local `dofs` to Trefftz DG



Space decomposition $V_h = \mathbb{P}^k(\mathcal{T}_h) = \mathbb{T}_h \oplus \mathbb{L}_h$ with (for a suitable space Q_h)

- $\mathcal{L} u_{\mathbb{T}} = 0$ (in Q_h') for all $u_{\mathbb{T}} \in \mathbb{T}_h$ ((classical) Trefftz functions)
- $\mathcal{L} : \mathbb{L}_h \to Q_h'$ bijective (local functions, solve for inhomogeneous r.h.s.)

[11] A. Lozinski. *A primal [dG] method with static condensation on very general meshes.*, Numerische Mathematik, 2019

[12] C.L., P. Stocker, I. Voulis. *A unified framework for Trefftz-like discretization methods*, in preparation, 2024

Example: Trefftz DG for Poisson

$$\mathcal{L}u = -\Delta u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

DG discretization

Find $u_h \in \mathbb{P}^k(\mathcal{T}_h)$, s.t. $a_h(u_h, v_h) = \ell_h(v_h) \qquad \forall v_h \in \mathbb{P}^k(\mathcal{T}_h)$ with

$$a_h(u,v) = \sum_K \int_K \nabla u \nabla v \, dx + \sum_{F \in \mathcal{F}_h^{\text{int}}} \int_F -\{\!\{\partial_{\mathbf{n}} u\}\!\} [\![v]\!] - \{\!\{\partial_{\mathbf{n}} v\}\!\} [\![u]\!] + \alpha k^2 h^{-1} [\![u]\!][\![v]\!] \, ds + bnd.$$

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Trefftz DG discretization with $\mathbb{P}^k = \mathbb{T}_h \oplus \mathbb{L}_h$, $Q_h = \mathbb{P}^{k-2} = -\Delta \mathbb{P}^k$

We search for $u_h \in \mathbb{P}^k(\mathcal{T}_h)$ with $u_h = u_{\mathbb{T}} + u_{\mathbb{L}}$, where

$$u_{\mathbb{T}} \in \mathbb{T}_h := \{v \in \mathbb{P}^k(\mathcal{T}_h),\ -\Delta v = 0 \text{ in } Q_h'\} = \{v \in \mathbb{P}^k(\mathcal{T}_h),\ -\Delta v = 0 \text{ on each } K \in \mathcal{T}_h\} \subset \mathbb{P}^k(\mathcal{T}_h),$$

$u_{\mathbb{L}} \in \mathbb{L}_h$: complementary space to $\mathbb{T}_h$ with $-\Delta : \mathbb{L}_h \to \mathbb{P}^{k-2} = Q_h'$ bijective (element-wise).

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$u_{\mathbb{L}} \in \mathbb{L}_h :$ complementary space to $\mathbb{T}_h$ with $-\Delta : \mathbb{L}_h \to \mathbb{P}^{k-2} = Q_h'$ bijective (element-wise).

We set $u_{\mathbb{L}} = (-\Delta|_{\mathbb{L}_h \to Q_h'})^{-1}(\Pi^{k-2} f)$ and solve the remaining homogenized Trefftz problem:

$$\text{Find } u_{\mathbb{T}} \in \mathbb{T}_h, \text{ s.t. } a_h(u_{\mathbb{T}}, v_{\mathbb{T}}) = \ell_h(v_{\mathbb{T}}) - a_h(u_{\mathbb{L}}, v_{\mathbb{T}}) \qquad \forall v_{\mathbb{T}} \in \mathbb{T}_h.$$

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[12] C.L., P. Stocker, I. Voulis. *A unified framework for Trefftz-like discretization methods*, in preparation, 2024

Rewrite as Block System with a Local-Global splitting

$$\begin{pmatrix} A_{\mathbb{TT}} & A_{\mathbb{TL}} \\ A_{\mathbb{LT}} & A_{\mathbb{LL}} \end{pmatrix} \cdot \begin{pmatrix} u_{\mathbb{T}} \\ u_{\mathbb{L}} \end{pmatrix} = \begin{pmatrix} \ell_{\mathbb{T}} \\ \ell_{\mathbb{L}} \end{pmatrix}$$

with

- $A_{\mathbb{LT}} = 0$ (globally coupled part does not influence local solution)
- $A_{\mathbb{LL}} = -\Delta|_{\mathbb{L}_h \to Q_h'}$ (element-wise decoupled, can be solved for in parallel)
- $\ell_{\mathbb{L}} = \Pi_{Q_h} f$
- $A_{\mathbb{TT}}$, $A_{\mathbb{TL}}$ and $\ell_{\mathbb{T}}$ are the globally coupled DG parts.
- Large linear systems only have to be solved for $\mathbb{T}_h$ part ($\mathcal{O}(k^{d-1})$`dofs`)
- Implementation of local solutions can reuse local SVD decomposition of $\mathbf{W}$ ($w_h(\cdot,\cdot)$)

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$$\begin{pmatrix} A_{\mathbb{T}\mathbb{T}} & A_{\mathbb{T}\mathbb{L}} \\ A_{\mathbb{L}\mathbb{T}} & A_{\mathbb{L}\mathbb{L}} \end{pmatrix} \cdot \begin{pmatrix} u_{\mathbb{T}} \\ u_{\mathbb{L}} \end{pmatrix} = \begin{pmatrix} \ell_{\mathbb{T}} \\ \ell_{\mathbb{L}} \end{pmatrix}$$

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$\rightsquigarrow$ Implementation for inhomogeneous problems ✅

[12] C.L., P. Stocker, I. Voulis. *A unified framework for Trefftz-like discretization methods*, in preparation, 2024

Non-polynomial Trefftz spaces

Many problems don't have suitable polynomial Trefftz spaces

Examples: $\mathcal{L} = -\Delta \pm \mathrm{Id}$, $\quad \mathcal{L} = -\Delta + b \cdot \nabla$, $\quad \mathcal{L} = -\mathrm{div}(\alpha \nabla \cdot)$, α not constant

[13] C. J. Gittelson, R. Hiptmair, and I. Perugia, *Plane wave [dG] methods: Analysis of the h-version*, ESAIM:M2AN, 2009

[14] L.-M. Imbert-Gérard, A. Moiola, P. Stocker, *A space-time quasi-Trefftz DG [...]*, Math. Comp., 2023

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How to do Trefftz in these cases?[13,14]

Weak Trefftz condition (the L^2 version)[10]

Now, we relax the condition by choice of Q_h ($Q_h = \mathcal{L}\mathbb{P}^k(\mathcal{T}_h)$ recovers "strong" Trefftz).

$$\implies \text{Weak Trefftz space:} \quad \mathbb{T}^k(\mathcal{T}_h) = \{ v \in \mathbb{P}^k(\mathcal{T}_h) \mid \Pi_{Q_h} \mathcal{L} v = 0 \}$$

with Π_{Q_h} the L^2 projection into Q_h.

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$\rightsquigarrow$ Implementation for non-polynomial Trefftz spaces ✅

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Stability analysis of Embedded Trefftz DG

We solve for $u_h = u_{\mathbb{T}} + u_{\mathbb{L}}$: Find $u_h \in V_h$ s.t.

$$B_h(u_h, (q_h, v_{\mathbb{T}})) := \underbrace{\langle \mathcal{L} u_h, q_h \rangle}_{\langle \cancel{\mathcal{L} u_{\mathbb{T}} +} \mathcal{L} u_{\mathbb{L}}, q_h \rangle} + a_h(u_h, v_{\mathbb{T}}) = \langle f, q_h \rangle + \ell_h(v_h) \quad \forall\ (q_h, v_h) \in Q_h \times \mathbb{T}_h.$$

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Assume:

- local stability: $\|\mathcal{L} u_{\mathbb{L}}\|_{Q_h'} \simeq \|u_{\mathbb{L}}\|_h,\ u_{\mathbb{L}} \in \mathbb{L}_h$
- coercivity[15] on $\mathbb{T}_h$: $a_h(u_{\mathbb{T}}, u_{\mathbb{T}}) \geq \alpha \|u_{\mathbb{T}}\|_h^2,\ u_{\mathbb{T}} \in \mathbb{T}_h$
- continuity: $a_h(u_{\mathbb{L}}, u_{\mathbb{T}}) \leq \tilde{\beta} \|u_{\mathbb{L}}\|_h \|u_{\mathbb{T}}\|_h \leq \beta \|\mathcal{L} u_{\mathbb{L}}\|_{Q_h'} \|u_{\mathbb{T}}\|_h$

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Choosing $q_h = \gamma R_{Q_h} \mathcal{L} u_{\mathbb{L}}$ and $v_{\mathbb{T}} = u_{\mathbb{T}}$ yields

$$\begin{aligned} B_h(u_h, (q_h, v_{\mathbb{T}})) &= B_h(u_h, (\gamma R_{Q_h} \mathcal{L} u_{\mathbb{L}}, u_{\mathbb{T}})) = \gamma \langle \mathcal{L} u_{\mathbb{L}}, R_{Q_h} \mathcal{L} u_{\mathbb{L}} \rangle + a_h(u_{\mathbb{T}}, u_{\mathbb{T}}) + a_h(u_{\mathbb{L}}, u_{\mathbb{T}}) \\ &\geq \gamma \|\mathcal{L} u_{\mathbb{L}}\|_{Q_h'}^2 + \alpha \|u_{\mathbb{T}}\|_h^2 - \beta \|u_{\mathbb{T}}\|_h \|\mathcal{L} u_{\mathbb{L}}\|_{Q_h'} \geq (\gamma - \frac{\beta^2}{2\alpha}) \|\mathcal{L} u_{\mathbb{L}}\|_{Q_h'}^2 + \frac{\alpha}{2} \|u_{\mathbb{T}}\|_h^2 \\ &\gtrsim \|\mathcal{L} u_{\mathbb{L}}\|_{Q_h'}^2 + \|u_{\mathbb{T}}\|_h^2 \gtrsim \underbrace{\left(\|\mathcal{L} u_{\mathbb{L}}\|_{Q_h'}^2 + \|u_{\mathbb{T}}\|_h^2 \right)^{\frac{1}{2}}}_{\simeq \|u_h\|_h} \|(q_h, v_{\mathbb{T}})\|_{Q_h \times \mathbb{T}_h} \end{aligned}$$

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[12] C.L., P. Stocker, I. Voulis. *A unified framework for Trefftz-like discretization methods*, in preparation, 2024

[15] inf-sup stability suffices

A priori error estimates of Embedded Trefftz DG

Classical Trefftz DG[10,16]

Classical Trefftz DG estimates are of the form

$$\|u - u_h\|_h \lesssim \inf_{v_h \in \mathbb{T}_h + u_h^{\mathrm{p}}} \|u - v_h\|_h$$

with $u_h^{\mathrm{p}} \in V_h$ a particular solution to $\mathcal{L} u_h \approx f$ and the need to construct a suitable interpolation in the Trefftz space (often averaged Taylor polynomials).

Theorem[12]

From stability (last slide), consistency and continuity we obtain (Céa):

$$\|u - u_h\|_h \lesssim \inf_{v_h \in V_h} \|u - v_h\|_h$$

Method exploits local/global splitting, but approximation problem is on the whole space.

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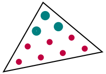
[10] C.L., P. Stocker. *Embedded Trefftz Discontinuous Galerkin methods*, IJNME, 2022

[12] C.L., P. Stocker, I. Voulis. *A unified framework for Trefftz-like discretization methods*, in preparation, 2024

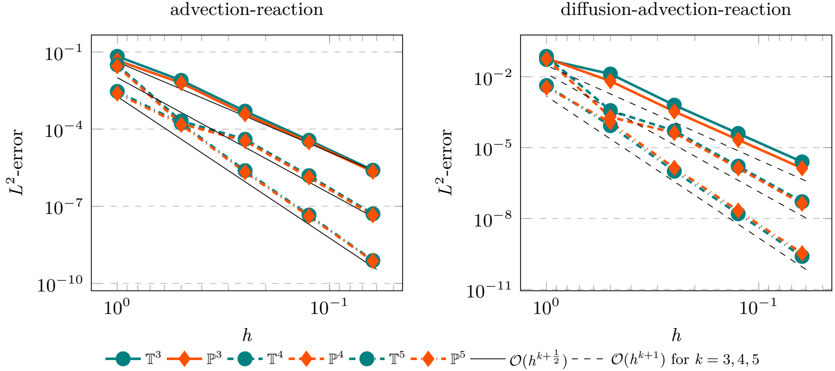
[16] C.L., P. L. Lederer, P. Stocker, *Trefftz [DG] discretization for the Stokes problem*, Numerische Mathematik, 2024

Trefftz DG for scalar PDEs

$\beta \cdot \nabla u + \gamma u = f \quad \text{in } \Omega \quad \rightsquigarrow$ "prototype" operator $\quad \bar{\mathcal{L}}_K = \bar{\beta}_K \cdot \nabla u$



$-\operatorname{div}(\alpha \nabla u) + (\beta \cdot \nabla) u + \gamma u = f \quad \text{in } \Omega \quad \rightsquigarrow$ "prototype" operator $\quad \bar{\mathcal{L}}_K = -\bar{\alpha}_K \Delta u$



We obtain optimal a priori error estimates (as for DG)

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Outline

🏁 Summary

🛠️ Embedded/Weak Trefftz DG as a general purpose discretization:

- 🔵 inhomogeneous r.h.s. ✅
- 🟢 PDE operators with non-polynomial kernels (e.g. non-constant coefficients) ✅
- 🟠 Vectorial and mixed PDE problems (Stokes ✅)

💪 Computational costs (sparsity) similar to Hybrid DG

- 🤝 Combines naturally with some stabilizations (e.g. 👻-penalty)
- *️⃣ Allows reducing FE space locally w.r.t. other constraints (tang.)

📉 Advantages for polytopal meshes

🌅 Outlook

- 🕒 time-dependent PDEs (with time-stepping; e.g. Navier-Stokes)
- 🔄 nonlinear PDEs (stationary Navier-Stokes) ✍️
- ⛓️ conforming Trefftz methods 🚧
  (generic basis fcts. through "moments" and kernel property)
  - 🟢 Implementation of C^1 elements (e.g. Argyris) in a generic framework
  - 🟣 (relaxed) $H(\text{div})$-conforming Stokes-Trefftz
  - 🔵 H^1 elements with local bubbles "eliminated the Trefftz way"

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Thank you for your attention!

Back-Up Slides

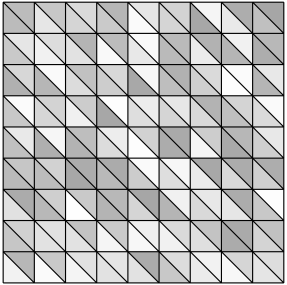
- Comparison algorithmic complexity: DG–HDG–Trefftz DG
- Sparsity comparison on polytopal meshes
- Trefftz DG for Stokes

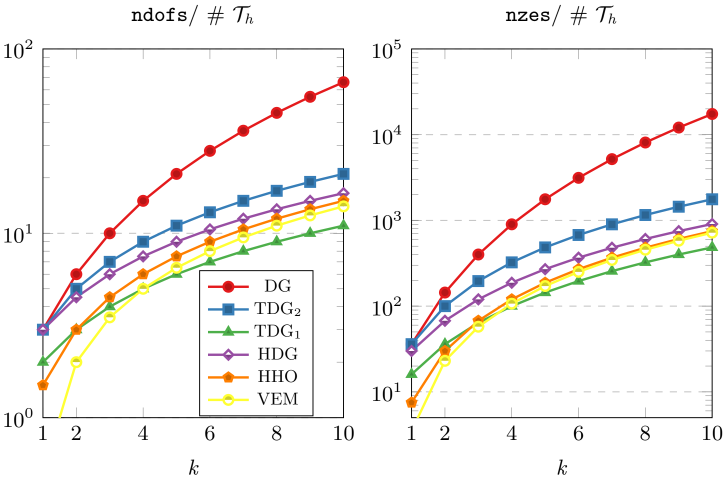
Algorithmic complexity: A rough comparison

- direct solver
- $N_{\text{el}} := \#\mathcal{T}_h \sim h^{-d}$
- k-scaling (no constants)

Costs:	Standard DG	Trefftz DG	Embedded Trefftz DG	Hybrid DG
Vector representation:				
total `ndofs` stored	$\sim N_{\text{el}} k^d$	$\sim N_{\text{el}} k^{d-1}$	$\sim N_{\text{el}} k^d$	$\sim N_{\text{el}} k^d$
globally coupled `ndofs`	$\sim N_{\text{el}} k^d$	$\sim N_{\text{el}} k^{d-1}$	$\sim N_{\text{el}} k^{d-1}$	$\sim N_{\text{el}} k^{d-1}$
Additional costs:			Setup $\mathbf{T}$:	static cond.:
	—	—	$\sim N_{\text{el}} k^{3d}$	$\sim N_{\text{el}} k^{3d}$
Global linear systems:				
global matrix	$\mathbf{A}$	$\mathbf{A}$	$\mathbf{T}^T\mathbf{A}\mathbf{T}$	$\mathbf{S}$
`nzes`	$\sim N_{\text{el}} k^{2d}$	$\sim N_{\text{el}} k^{2d-2}$	$\sim N_{\text{el}} k^{2d-2}$	$\sim N_{\text{el}} k^{2d-2}$

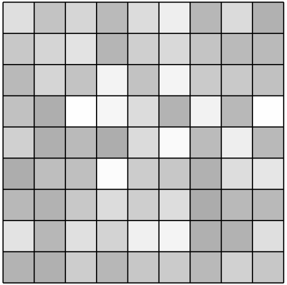
Sparsity comparison: on (periodic) triangles

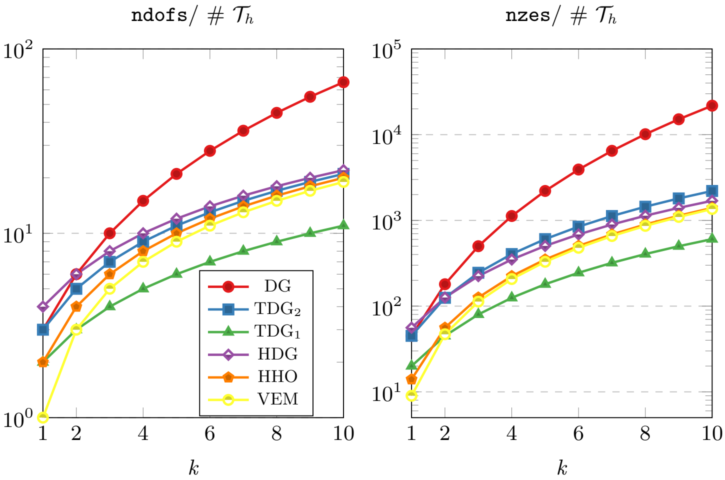




[17] C.L., P. Stocker, M. Zienecker, *Sparsity comparison of polytopal finite element methods*, PAMM 2024

Sparsity comparison: on (periodic) quadrilaterals





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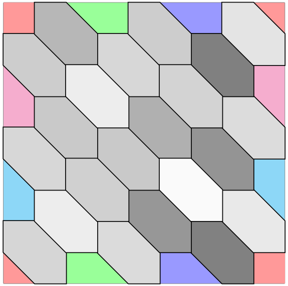
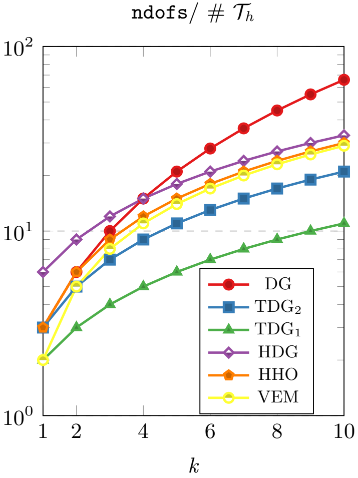
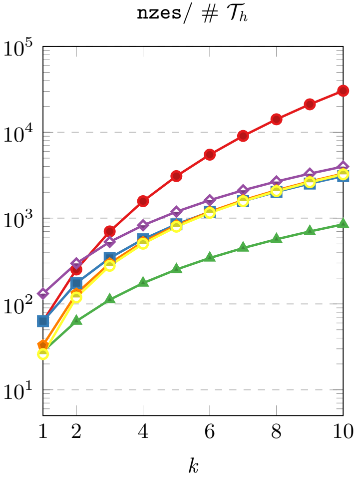
Sparsity comparison: on (periodic) hexagons







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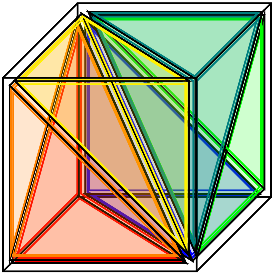
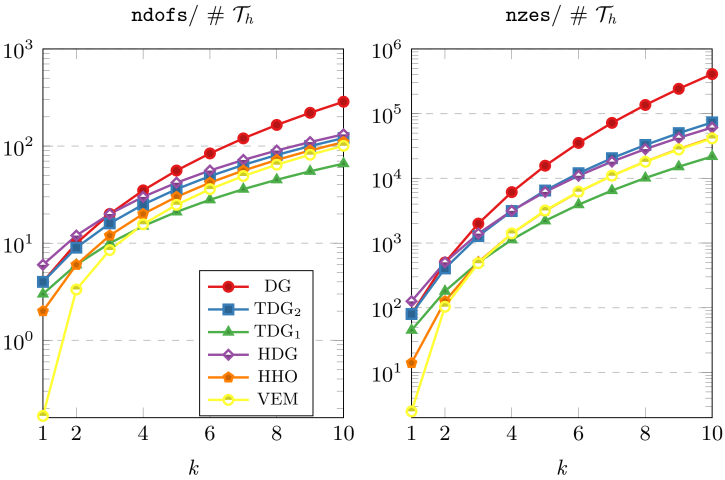
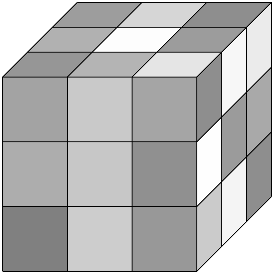
Sparsity comparison: on (periodic) tetrahedra

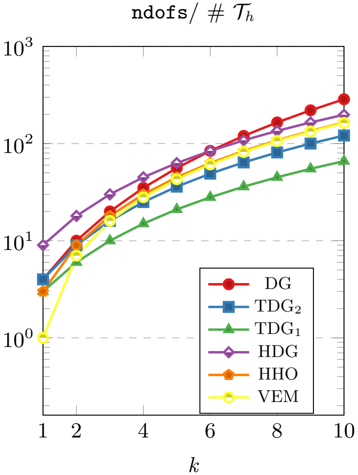


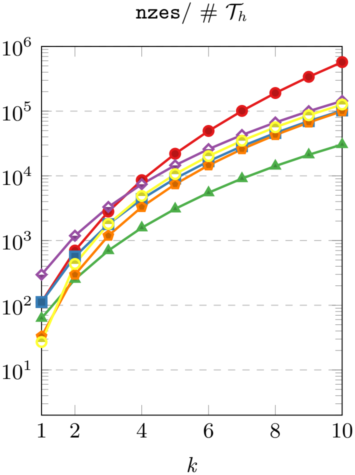


[17] C.L., P. Stocker, M. Zienecker, *Sparsity comparison of polytopal finite element methods*, PAMM 2024

Sparsity comparison: on (periodic) hexahedra

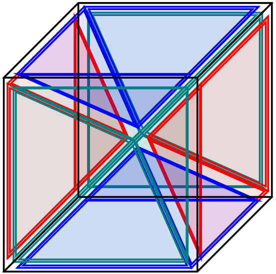


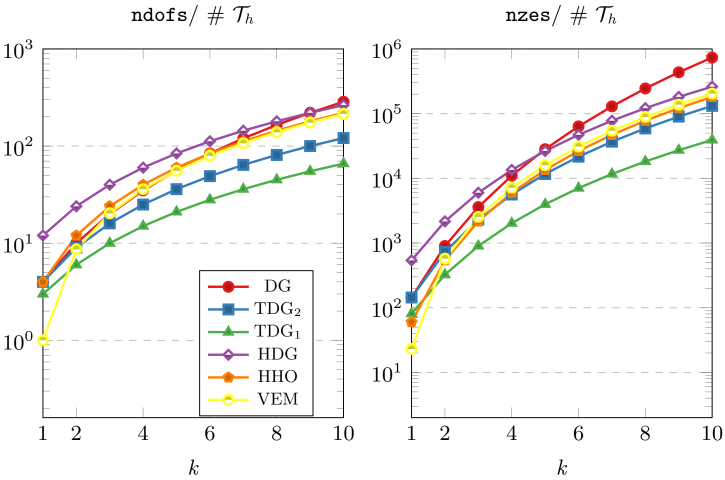




[17] C.L., P. Stocker, M. Zienecker, *Sparsity comparison of polytopal finite element methods*, PAMM 2024

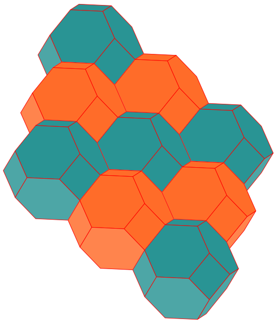
Sparsity comparison: on (periodic) octahedra

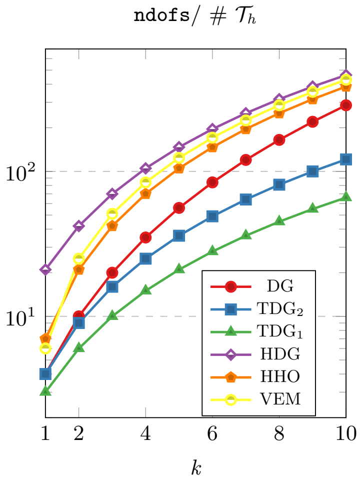


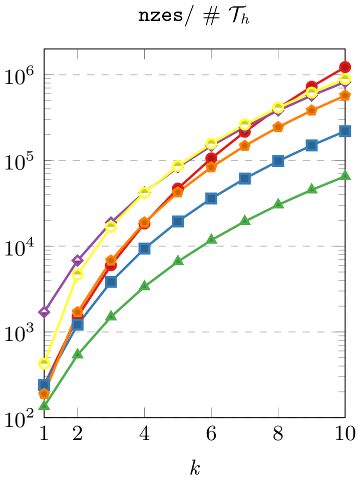


[17] C.L., P. Stocker, M. Zienecker, *Sparsity comparison of polytopal finite element methods*, PAMM 2024

Sparsity comparison: on (periodic) truncated octahedra

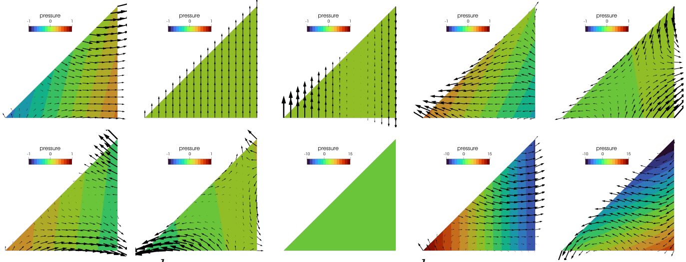






[17] C.L., P. Stocker, M. Zienecker, *Sparsity comparison of polytopal finite element methods*, PAMM 2024

Trefftz DG for Stokes

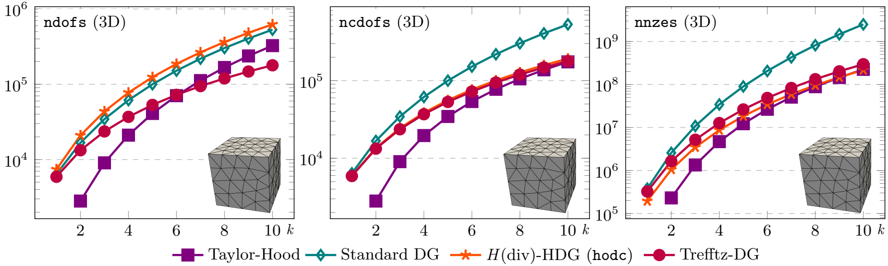


$$-\nu \Delta u + \nabla p = f \quad \text{in } \Omega,$$
$$-\mathrm{div} u = g \quad \text{in } \Omega.$$

$$\boldsymbol{X}_h^k(\mathcal{T}_h) = [\mathbb{P}^k(\mathcal{T}_h)]^d \times \mathbb{P}^{k-1}(\mathcal{T}_h) \backslash \mathbb{R}$$

$$\mathbb{T}^k(\mathcal{T}_h) := \{(u_h, p_h) \in \boldsymbol{X}_h^k \mid \mathcal{L}(u_h, p_h) = 0\},$$

with $\mathcal{L} : \boldsymbol{X}_h^k \to Q_h', (u_h, p_h) \mapsto (-\Delta u_h + \nabla p_h, -\mathrm{div} u_h)$



[16] C.L., P. L. Lederer, P. Stocker, *Trefftz [DG] discretization for the Stokes problem*, Numerische Mathematik, 2024