Time discretization with unfitted finite element methods for moving domain problems

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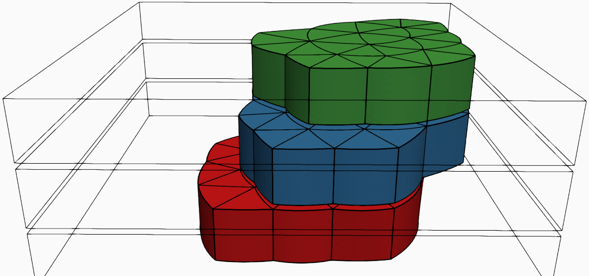
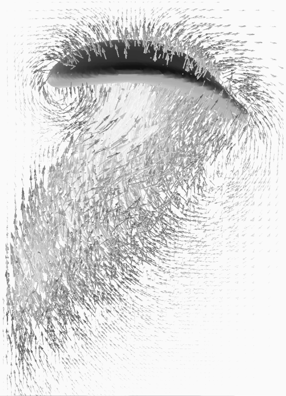
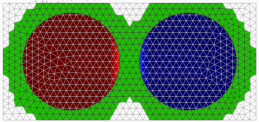
[6]University of Jena

Outline

- Motivation: unfitted FEM
- Time integration for unfitted FEM
- Approach 1: Eulerian Time Stepping
- Approach 2: Unfitted Space-Time FEM
    - Numerical integration
    - Numerical examples

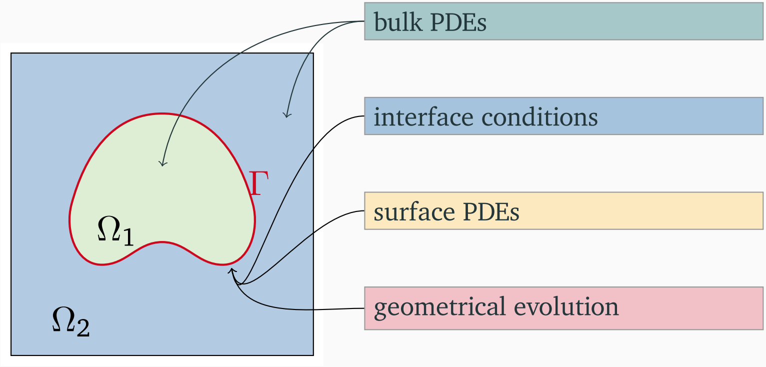




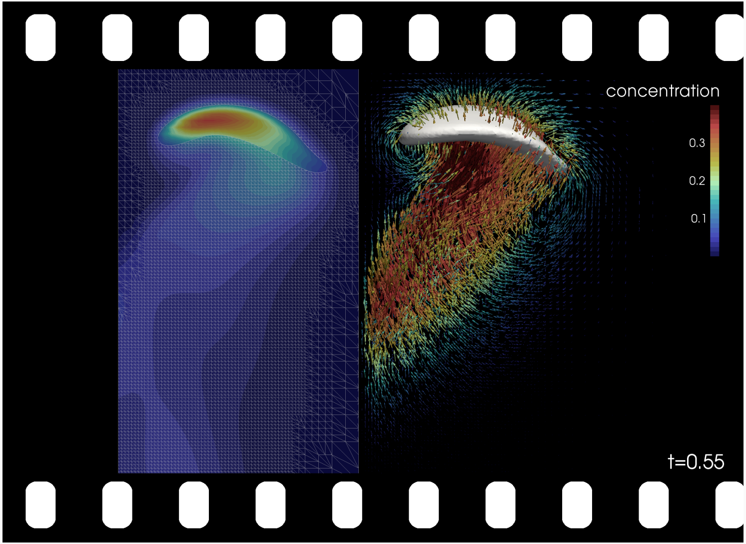


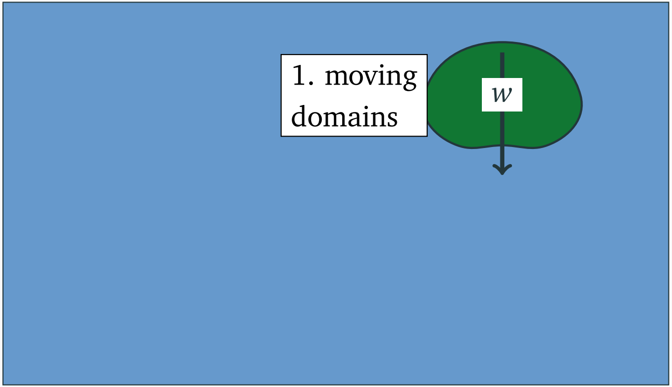
Motivation: unfitted finite element methods
What is it ? Why is it interesting ?

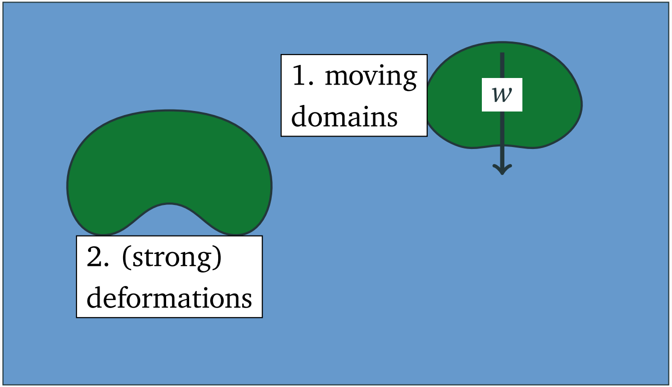
Motivation: Two phase flow problems (e.g. oil droplet in water)

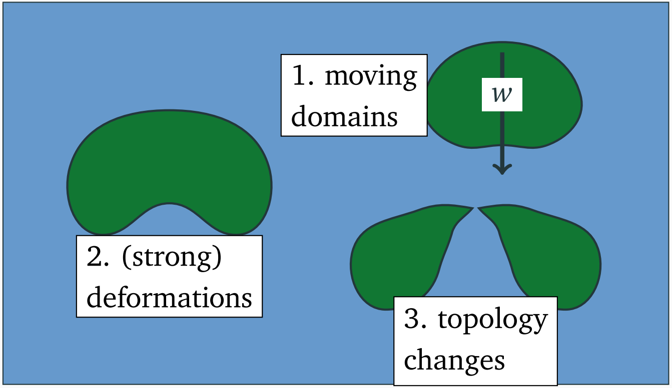


- evolution of **complex geometry**
- sub-problems are coupled (**nonlinear**)

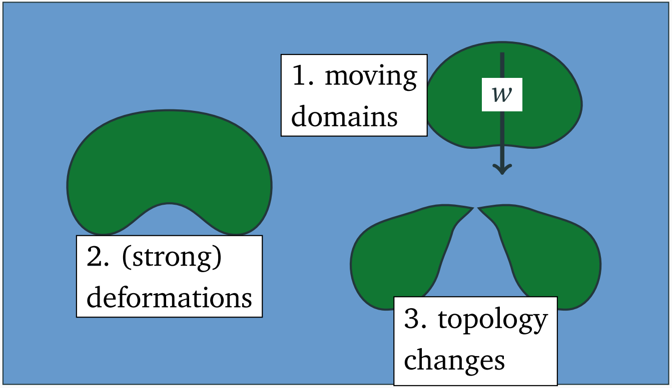

concentration
0.3
0.2
0.1
t=0.55


1. moving domains
w

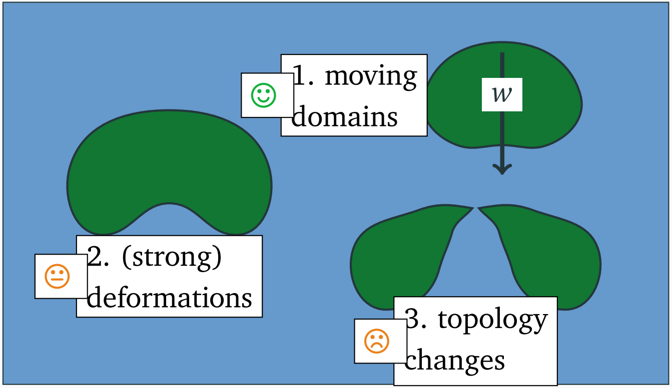

1. moving domains
w
2. (strong) deformations



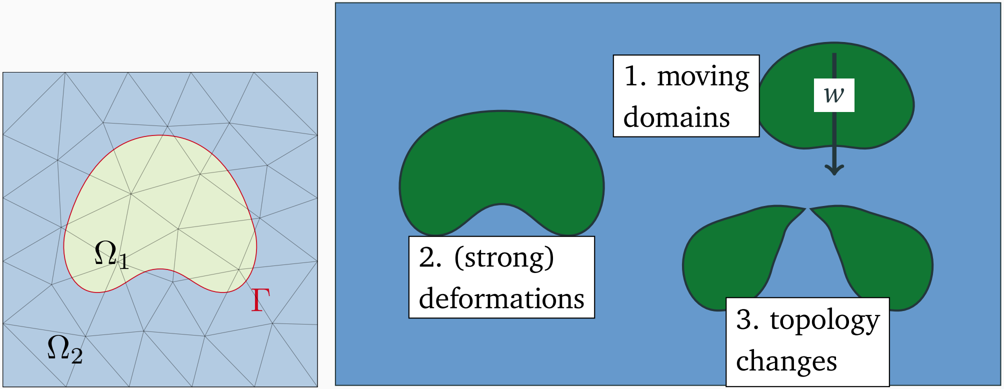
1. moving domains
w
2. (strong) deformations
3. topology changes



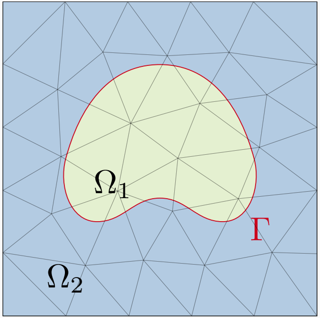
With fitted meshes?

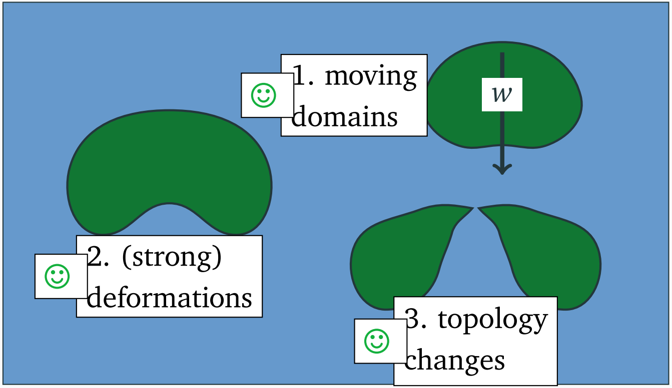


With fitted meshes? $\rightsquigarrow$ possible, but cumbersome



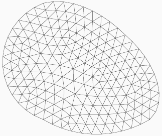
Idea of (geometrically) unfitted discretizations:
remove burden of fitted meshes (generation/tracking/remeshing) by decoupling mesh and geometry (e.g. level set)



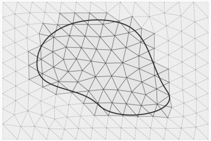


Idea of (geometrically) unfitted discretizations:
remove burden of fitted meshes (generation/tracking/remeshing)
by decoupling mesh and geometry (e.g. level set) $\rightsquigarrow$ flexible geometry handling,
New challenges: shape irregular cuts, numerical integration, time integration

Rough Comparison: fitted and unfitted discretizations



- results for small/no deformation:
  - high order accuracy
  - efficient linear solver concepts
  - robust implementation
  - rigorous error analysis, ...



⚠️ important key properties of standard FEM have to be re-established:

- impl. of boundary conditions
- robust numerical integration
- linear solver concepts
- stable (time) discretization

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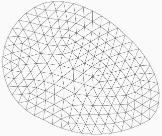

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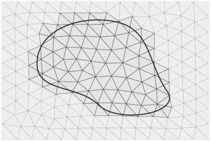
⚠ For strong deformation or topology changes this is much harder.



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- suitable for strong deform./ topo. changes

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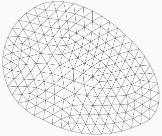

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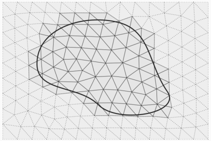
- Often mesh generation for complex geometries is very expensive



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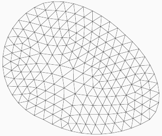

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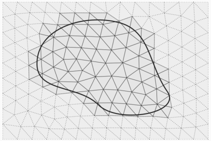

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- impl. of boundary conditions
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- stable (time) discretization

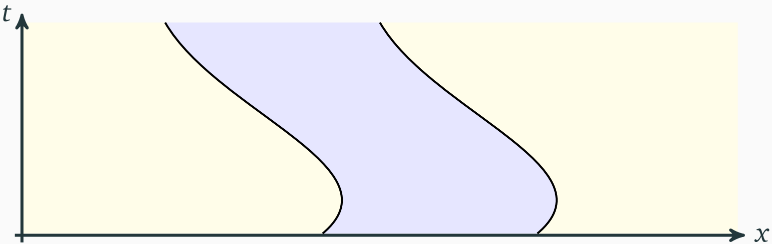
- suitable for strong deform./ topo. changes
- mesh generation "for free"

⇝ Assumption: we are convinced that going unfitted is an interesting idea.

How to make it work?

Time integration for unfitted FEM
Why is time integration an issue?

Time discretization for problems on evolving geometries



[1] P. Hansbo, M. Larson, S. Zahedi. Characteristic Cut FEM for Convection-Diffusion Problems on Time Dependent Surfaces, CMAME, 2015
[2] Q.Z. Chuwen Ma, W, Zhen. A high-order fictitious-domain method for the adv.-diff. eq. on time-varying dom., arXiv:2104.01870, 2021
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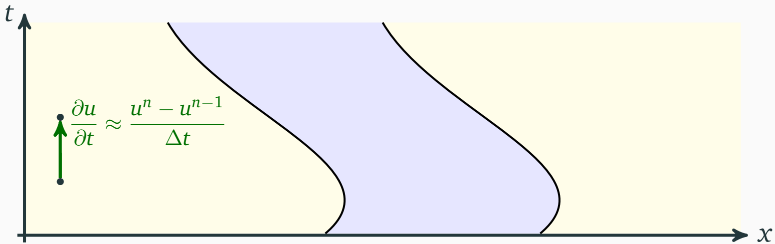

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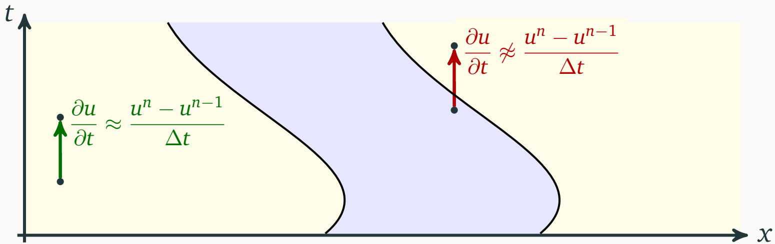

Naive method of lines is not applicable!

Alternatives:

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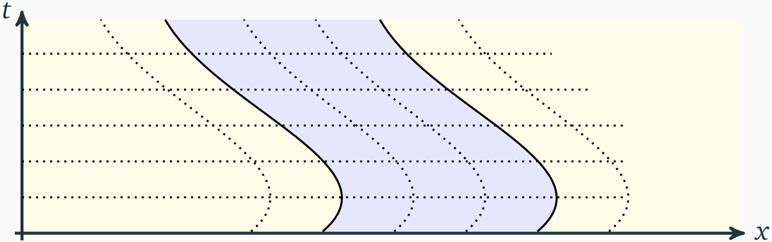

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- Let mesh follow the geometry (Lagrangian view point)

Alternatives:

✗ framework

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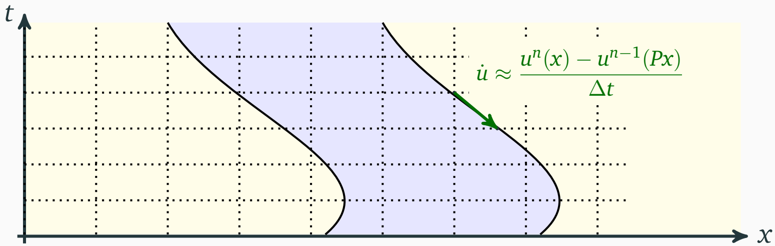

Naive method of lines is not applicable! unfitted Alternatives:

- Use *characteristics* $\dot{u} = \frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u$ [1,2] (✗) (framework)

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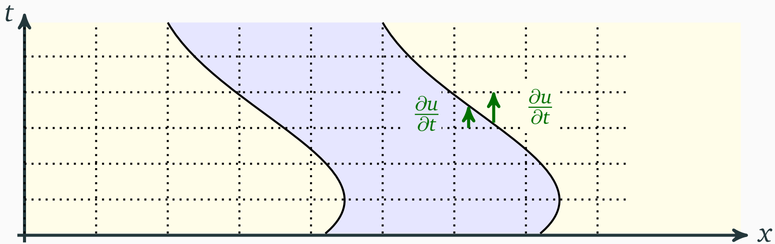

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- Use *characteristics* $\dot{u} = \frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u$ [1,2] (✗) (framework)
- Separate domains via a *space-time* formulation[3] ✓

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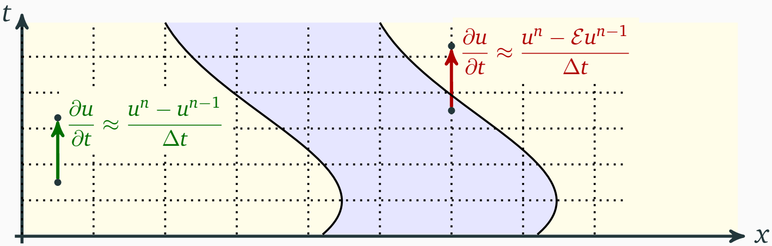

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- Separate domains via a *space-time* formulation[3] ✓
- Extend solutions to neighborhood[4] ✓

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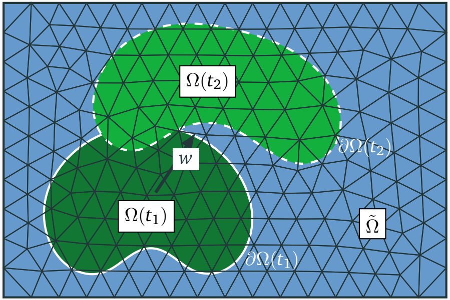
Model problem for the remainder (single phase)



$$
\begin{aligned}
\partial_t u - \Delta u + \mathbf{w} \cdot \nabla u &= f && \text{in } \Omega(t), && t \in [0, T], \\
\nabla \cdot \mathbf{w} &= 0 && \text{in } \Omega(t), && t \in [0, T], \\
\nabla u \cdot \mathbf{n}_{\partial\Omega} &= 0 && \text{on } \partial\Omega(t), && t \in [0, T], \\
u(\cdot, t = 0) &= u_0 && \text{in } \Omega(t = 0).
\end{aligned}
$$

Approach 1:
Eulerian Time Stepping

Method of lines for unfitted FEM on moving domains

Implicit Euler without care

$$\frac{u^n - \quad u^{n-1}}{\Delta t} + w^n \cdot \nabla u^n - \alpha \Delta u^n = 0, \qquad \text{on} \;\; \Omega^n.$$

- semi-discrete (discrete in time):
  u^{n-1} may not be def. on Ω^n

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Method of lines for unfitted FEM on moving domains

Implicit Euler with more care

$$\frac{u^n - \mathcal{E}u^{n-1}}{\Delta t} + w^n \cdot \nabla u^n - \alpha \Delta u^n = 0, \qquad \text{on } \Omega^n.$$

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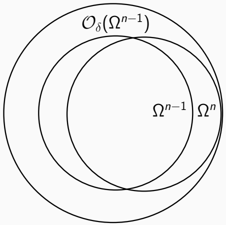

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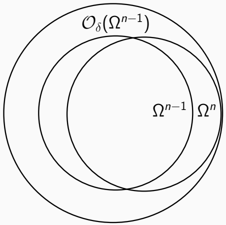
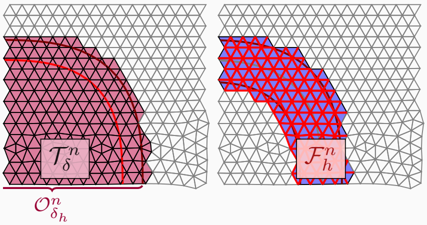
- fully discrete: Use 👻-penalty stabilization for discrete extension of FE functions





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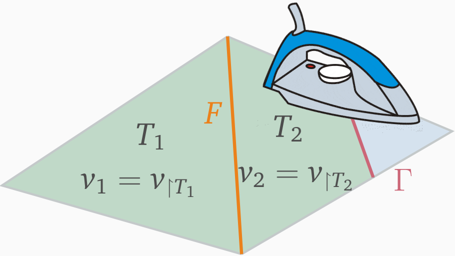
Facet 👻-penalty[5] on one facet (for extension domain / cut stabilization)

v: piecewise polynomial defined on macro element $\omega_F := T_1 \cup T_2$

v_i: restriction of v to T_i

$[\![v]\!]_{\omega_F}(x) := v_1(x) - v_2(x)$ for $x \in \omega_F$ defined through polynomial extension.



$\|v\|^2_{T_2} \leq C \left(\| [\![v]\!]_{\omega_F} \|^2_{T_2} + \|v\|^2_{T_1} \right)$: "control on T_2 if T_1 and $[\![v]\!]_{\omega_F}$ controlled[6]."

---

[5] E. Burman, Ghost penalty, Comptes Rendus Mathematique, 2010

[6] independent on varitational form on T_2 (cut configuration, etc..)

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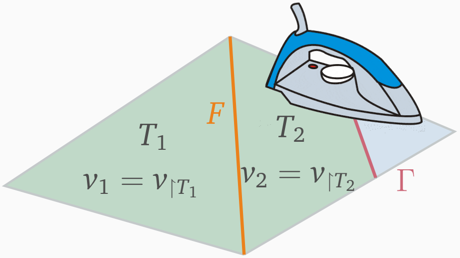

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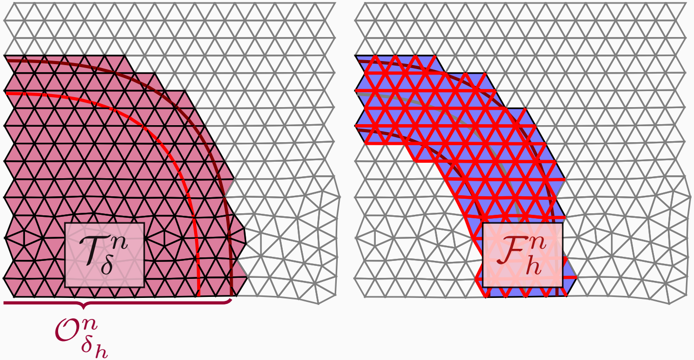
👻-penalty stabilization on one facet F: $+\int_{\omega_F} \frac{1}{h^2}[\![u]\!]_{\omega_F}^2 \, dx$

penalizes deviations from a patch-wise polynomial (i.e. h.o. discontinuities).

[5] E. Burman, Ghost penalty, Comptes Rendus Mathematique, 2010

[6] independent on varitational form on T_2 (cut configuration, etc.)

Implicit extension from $\mathcal{T}^n$ to $\mathcal{T}_\delta^n$



- stepping from t^{n-1} to t^n gives equations for unknowns in "active mesh" $\mathcal{T}^n$

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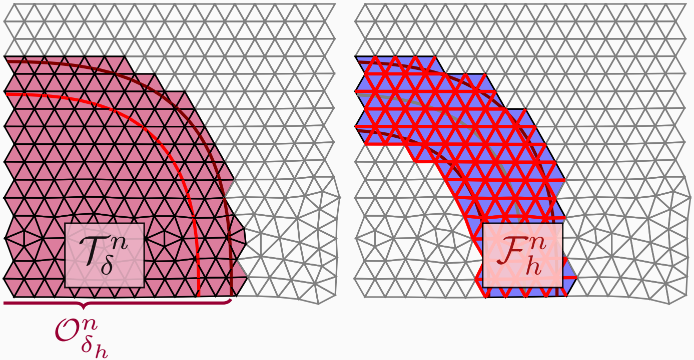

- stepping from t^{n-1} to t^n gives equations for unknowns in "active mesh" $\mathcal{T}^n$
- when solving for t^n add 👻-penalty stabilization in a δ-layer around $\partial\Omega_h^n$

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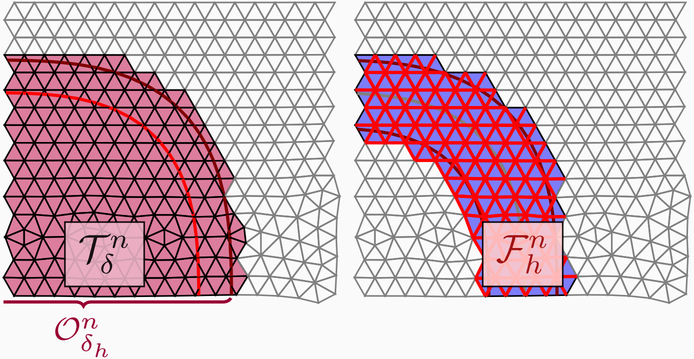
Implicit extension from $\mathcal{T}^n$ to $\mathcal{T}^n_\delta$



- stepping from t^{n-1} to t^n gives equations for unknowns in “active mesh” $\mathcal{T}^n$
- when solving for t^n add 👻-penalty stabilization in a δ-layer around $\partial\Omega^n_h$
- choose δ sufficiently large so that $\mathcal{T}^n_\delta \supset \Omega^{n+1}_h$

[4] C.L., M. Olshanskii. An Eulerian finite element method for PDEs in time-dependent domains. ESAIM: M2AN, 2019

Variational formulation of an implicit Euler step

$$\int_{\Omega_h^n} \frac{u_h^n - u_h^{n-1}}{\Delta t} v_h + \underbrace{a_h^n(u_h^n, v_h)}_{\text{convection diffusion bil. form}} + j_h^n(u_h^n, v_h) = 0 \quad \text{for all } v_h \in V_h^n (\text{FE Space}).$$

- 👻-penalty stabilization:

$$j_h^n(u,v) := \gamma_J \cdot \underbrace{\left(1 + \frac{\Delta t}{h}\right)}_{\text{anisotropy in space-time}} \sum_{F \in \mathcal{F}_R^{*,n}} \int_{\omega_F} \frac{1}{h^2} [\![u]\!]_{\omega_F} [\![v]\!]_{\omega_F} \, dx$$

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- Continuity of (implicit) extension (on V_h):

$$\|u_h\|^2_{\mathcal{O}_{\delta_h}(\Omega_h^n)} \leq (1+c\Delta t)\|u_h\|^2_{\Omega_h^n} + c\Delta t \|\nabla u_h\|^2_{\Omega_h^n} + c\Delta t \, j_h^n(u_h, u_h)$$

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- Provable convergence (here: implicit Euler)

$$\|u(t^n) - u_h^n\|_{\Omega_h^n} \lesssim \exp(ct_n) R(u) (\underbrace{\Delta t}_{\text{time}} + \underbrace{h^q}_{\text{geometry approx.}} + \underbrace{h^k}_{\text{space}} \cdot \underbrace{(1 + \Delta t / h)^{\frac{1}{2}}}_{\text{anisotropy in space-time}})$$

[4] C.L., M. Olshanskii. An Eulerian finite element method for PDEs in time-dependent domains. ESAIM: M2AN, 2019

Summary: Eulerian time stepping approach

- Simple: only requires spatial integrals / FE spaces

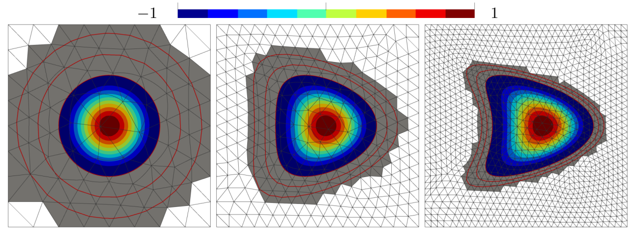
[7] Y. Lou, C.L.. Isoparametric unfitted BDF – finite element method for PDEs on evolving domains, SINUM 2022

[8] M. Olshanskii, H. v. Wahl. A conservative Eulerian finite element method for transport and diffusion in moving domains, arXiv 2024

[9] several technical details are skipped here...

Summary: Eulerian time stepping approach

- Simple: only requires spatial integrals / FE spaces
- higher order in time (and space[9]) possible with BDF-r schemes[7]



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Summary: Eulerian time stepping approach

- Simple: only requires spatial integrals / FE spaces
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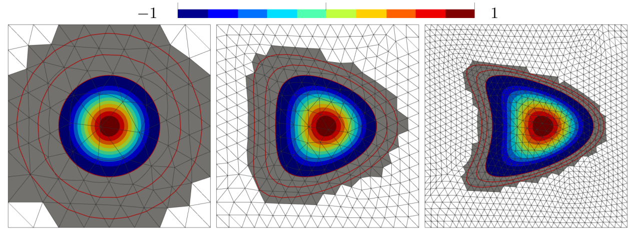


⚠ not robust for small diffusion (∇u needed to control extension)

⚠ not conservative (extension) $\rightsquigarrow$ exponential growth in estimates (Gronwall)

- conservative variant (first order only) without analysis[8]
- works even for topology changes

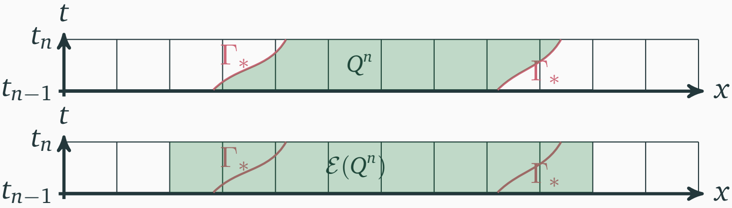
[7] Y. Lou, C.L.. Isoparametric unfitted BDF – finite element method for PDEs on evolving domains, SINUM 2022

[8] M. Olshanskii, H. v. Wahl. A conservative Eulerian finite element method for transport and diffusion in moving domains, arXiv 2024

[9] several technical details are skipped here...

Approach 2:
Unfitted Space-time FEM

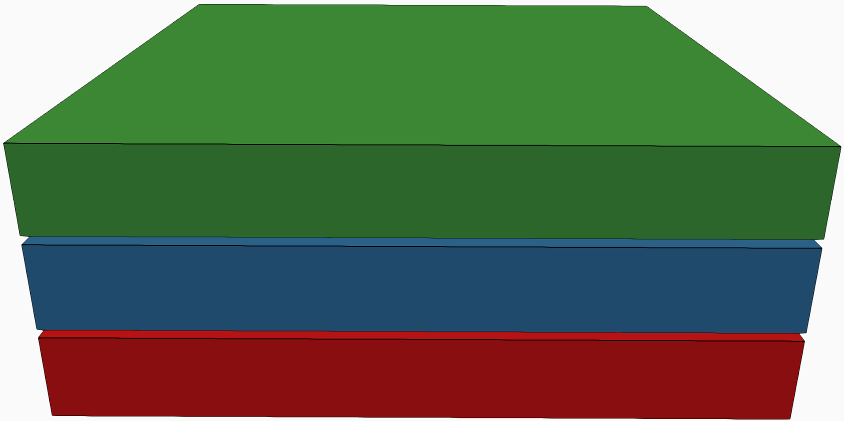
Space-time FE spaces using tensor product structure (for interior domain)



- Space-time prisms $Q_T^n = T \times I_n$ for $T \in \tilde{\mathcal{T}}_h$ ("active mesh")
- Extended time slab: $\mathcal{E}(Q^n)$
- Time slab FE space (global FE space discontinuous-in-time):

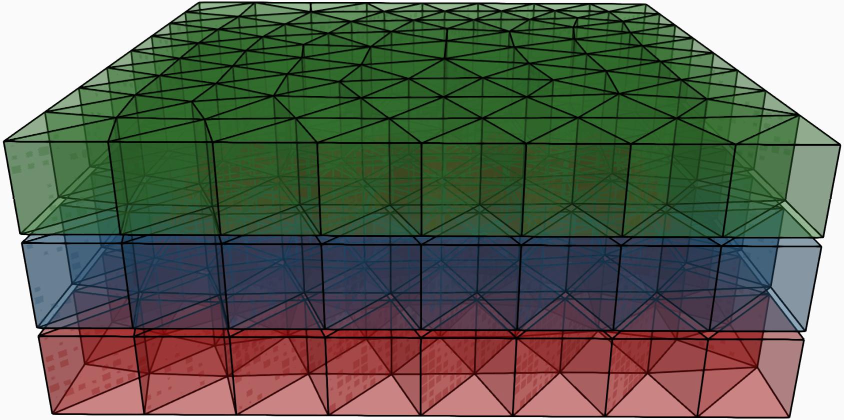
$$W_n := V_h^{k_s}(\mathcal{E}(\Omega^n)) \otimes \mathcal{P}^{k_t}((t_{n-1}, t_n))$$

Illustration: three time slabs



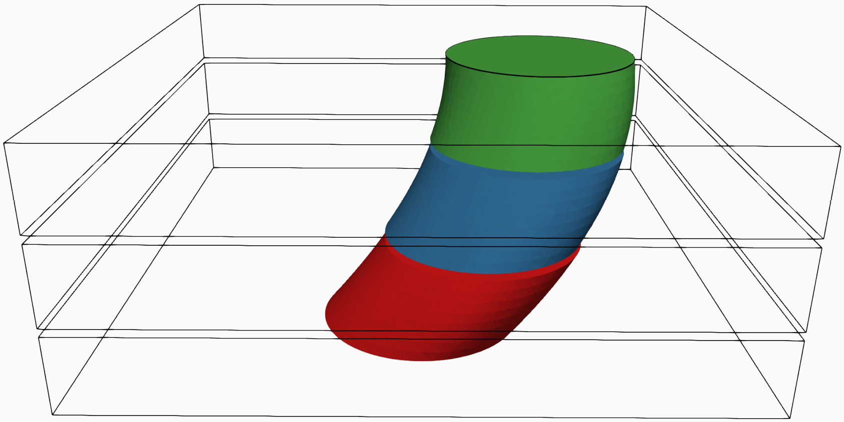
Space-Time slabs

Illustration: three time slabs



Space-Time slabs (tensor product mesh)

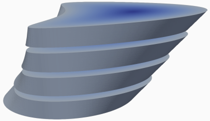
Illustration: three time slabs and an unfitted geometry



Space-Time level set domain $\phi < 0$

Space-Time approach 1: Discontinuous-in-time Galerkin scheme

Variational formulation on each time slab **(time stepping structure)**



Find $u_h \in W_n$ such that for all $v_h \in W_n$ holds:

$$(\partial_t u_h + \mathbf{w} \cdot \nabla u_h, v_h)_{Q^n} + (\nabla u_h, \nabla v_h)_{Q^n}$$
$$= (f, v_h)_{Q^n}$$

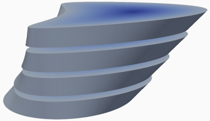
Contributions (1 as in fitted methods)

1 Consistency to PDE

$$\begin{aligned} \partial_t u - \Delta u + \mathbf{w} \cdot \nabla u &= f && \text{in } \Omega(t), && t \in [0, T], \\ \nabla u \cdot \mathbf{n}_{\partial\Omega} &= 0 && \text{on } \partial\Omega(t), && t \in [0, T], \end{aligned}$$

Space-Time approach 1: Discontinuous-in-time Galerkin scheme

Variational formulation on each time slab **(time stepping structure)**

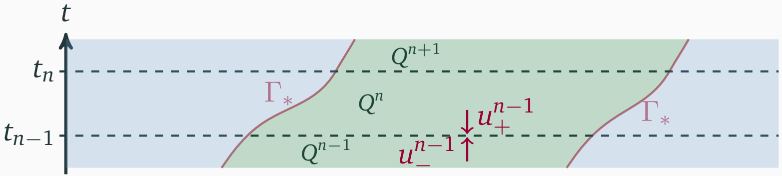


Find $u_h \in W_n$ such that for all $v_h \in W_n$ holds:

$$(\partial_t u_h + \mathbf{w} \cdot \nabla u_h, v_h)_{Q^n} + (\nabla u_h, \nabla v_h)_{Q^n} + \left(u_{h,+}^{n-1}, v_{h,+}^{n-1}\right)_{\Omega^{n-1}} = (f, v_h)_{Q^n} + \left(u_{h,-}^{n-1}, v_{h,+}^{n-1}\right)_{\Omega^{n-1}}.$$

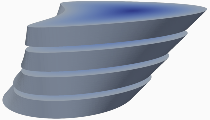
Contributions (1 & 2 as in fitted methods)

1. Consistency to PDE
2. Upwind stabilization



Space-Time approach 1: Discontinuous-in-time Galerkin scheme

Variational formulation on each time slab (time stepping structure)



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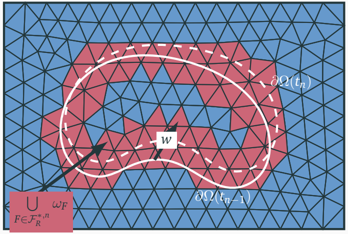
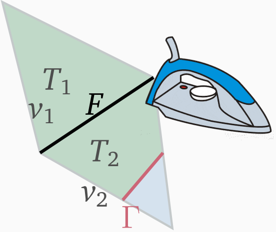
Contributions (1 & 2 as in fitted methods, 3 for unfitted methods)

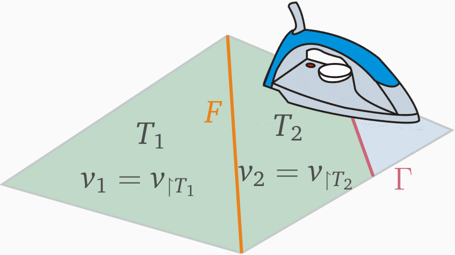
1. Consistency to PDE
2. Upwind stabilization
3. 👻-penalty stabilization
   - in the vicinity of cut prisms
   - "glues" polynomials together
   - re-enables inverse inequalities





[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

$[\![v]\!]_{\omega_F}(x,t) := v_1(x,t) - v_2(x,t)$ defined through polynomial extension in space.

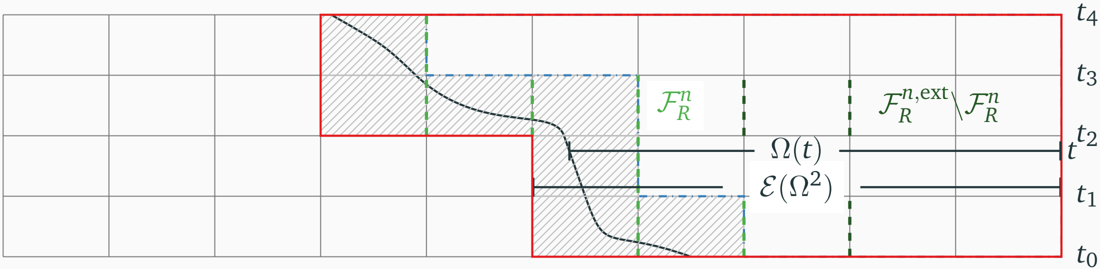


Anisotropic space-time stabilization:

$$j_h^n(u,v) := \int_{t_{n-1}}^{t_n} \gamma_J \cdot \left(1 + \frac{\Delta t}{h}\right) \sum_{F \in \mathcal{F}_R^{*,n}} \int_{\omega_F} \frac{1}{h^2} [\![u]\!]_{\omega_F} [\![v]\!]_{\omega_F} \, dx \, dt$$

[9] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

Stabilization (illustration for spatial 1D configuration)



$\bigcup_{n=1}^N \mathcal{E}(Q^n) \setminus \mathcal{I}(Q^n)$: cut elements

$\bigcup_{n=1}^N \mathcal{E}(Q^n)$: active elements

$\mathcal{F}_R^n(\mathcal{F}_R^{n,\mathrm{ext}})$: stabilization facets

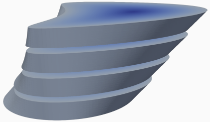
[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

Space-Time approach 2: Continuous-in-time Galerkin scheme

Variational formulation on each time slab **(time stepping structure)**

$$W_{n,0} := V_h^{k_s}(\mathcal{E}(\Omega^n)) \otimes \mathcal{P}_0^{k_t}((t_{n-1}, t_n)); \quad P_0^{k_t}((t_{n-1}, t_n))(t_{n-1}) = 0$$

$$V_n := V_h^{k_s}(\mathcal{E}(\Omega^n)) \otimes \mathcal{P}^{k_t-1}((t_{n-1}, t_n)), \quad u_{h,\text{init}} \in V_h^{k_s}(\mathcal{E}(\Omega^n)) \cdot \phi_0(t)$$



Find $u_h \in W_{n,\text{init}} := W_{n,0} + u_{h,\text{init}}$, s.t. for all $v_h \in V_n$:

[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

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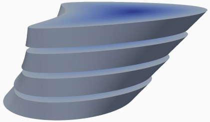

Contributions (1 as in fitted methods)

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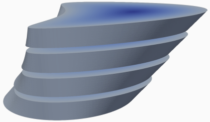

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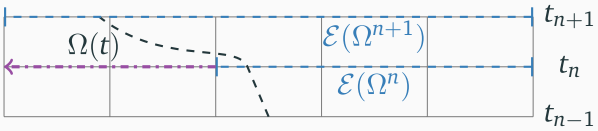
Contributions (1 as in fitted methods, 2 for unfitted methods)

1 Consistency to PDE

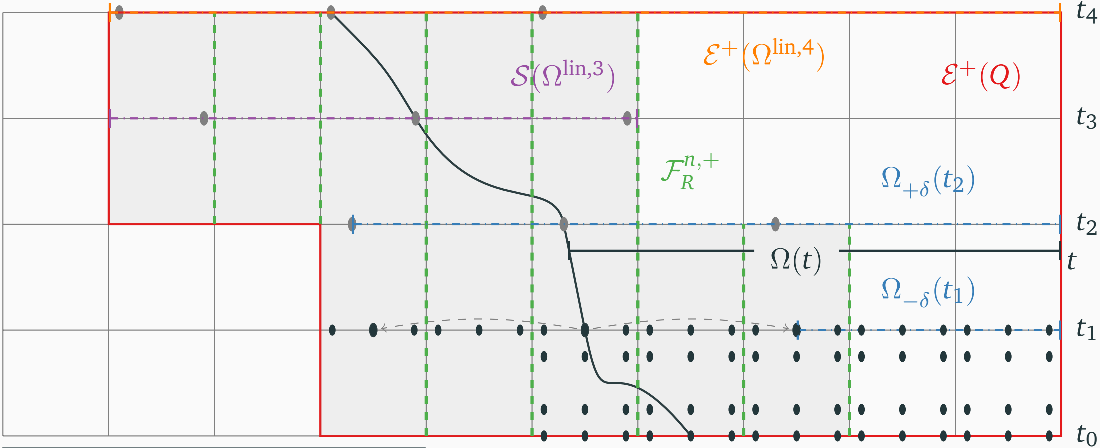
2 👻-penalty stabilization[5]

- as cut stabilization
- as extension mechanism at t_n (similar to Eulerian time stepping)



[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

Stabilization (illustration for spatial 1D configuration, $k_t = 3$)



10 F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

On the unfitted Space-time approaches

Lowest order ($k_t = 0$, DG)

- Yields conservative *Eulerian*-like method (no ext., space-time quadrature): one time `dof` per slab, find $u_h^n \in W_n = V_h^{k_s}(\mathcal{E}(\Omega^n)) \times \mathcal{P}^0$, s.t. $\forall \quad v_h \in W_n$:

$$(u_h^n, v_h)_{\Omega^n} + (u_h, -\mathbf{w} \cdot \nabla v_h)_{Q^n} + (\nabla u_h, \nabla v_h)_{Q^n} + j_h^n(u_h, v_h) \;=\; (f, v_h)_{Q^n} + \left(u_h^{n-1}, v_h\right)_{\Omega^{n-1}}.$$

On the unfitted Space-time approaches I/II

Lowest order ($k_t = 0$, DG)

- Yields conservative *Eulerian*-like method (no ext., space-time quadrature): one time dof per slab, find $u_h^n \in W_n = V_h^{k_s}(\mathcal{E}(\Omega^n)) \times \mathcal{P}^0$, s.t. $\forall \quad v_h \in W_n$:

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Linear-in-time ($k_t = 1$) – second order methods

- DG-in-time has two time dofs per step (superconvergence?)
- DG-in-time has provable higher order error bounds (not optimal analysis results though, see later)
- CG-in-time has one time dof per step
  - linear solver costs as for Eulerian time stepping or DG-0
  - no error analysis for CG-in-time, yet.

Higher order in time ($k_t > 1$)

- DG-in-time has $k_t + 1$ time `dofs` per step
- CG-in-time has k_t time `dofs` per step
- Higher order continuity in time can also be imposed:
  For $k_t \geq 3$ C^ℓ-continuity in time up to degree $\ell = \lfloor \frac{k_t - 1}{2} \rfloor$ can be imposed.
  (Galerkin-collocation methods; implementation similar to CG-in-time)
- Example GCC(3):
  $k_t = 3$, C^1-continuity in time and
  two time `dofs` per step (as many time `dofs` as DG with $k_t = 1$).

Theorem (assuming exact geometry handling)

Let $k_{\max} = \max\{k_s, k_t\}$. *There holds (u: exact solution,* u_h*: discrete solution):*

$$|||u - u_h||| \leq C\left(\Delta t^{k_t+1/2} + \left(1 + \frac{\Delta t}{h}\right)^{\frac{1}{2}} h^{k_s}\right) \|u\|_{H^{k_{\max}+2}(Q)},$$

with $|||u|||^2 := \sum_{n=1}^{N} \Delta t(\partial_t u, \partial_t u)_{Q^n} + [\![u]\!]^2 + (\nabla u, \nabla u)_Q$, *(*$[\![u]\!]$*: time jump/trace norm)*

Proof (sketch): Main idea follows along the line of standard Upwind DG analyses. □

Note: previous analyses[3,13] yield also higher order bounds (but weaker norms, different setting).

[11] C.L., A. Reusken. Analysis of a Nitsche XFEM-DG discretization for a class of two-phase mass transport problems. SINUM, 2013

[12] J. Preuß, Higher order unfitted isoparametric space-time FEM on moving domains, Ma. thesis, Univ. Göttingen, 2018

[13] S. Badia, H. Dilip, F. Verdugo, Space-time unfitted finite element methods for time-dependent problems on moving domain, CAMWA, 2023

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Proof (sketch): Main idea follows along the line of standard Upwind DG analyses. □

Note: previous analyses[3, 13] yield also higher order bounds (but weaker norms, different setting).

Formulation involves cut integrals. Higher order realization?

---

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[12] J. Preuß, Higher order unfitted isoparametric space-time FEM on moving domains, Ma. thesis, Univ. Göttingen, 2018

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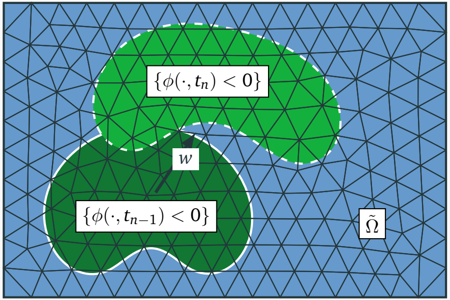
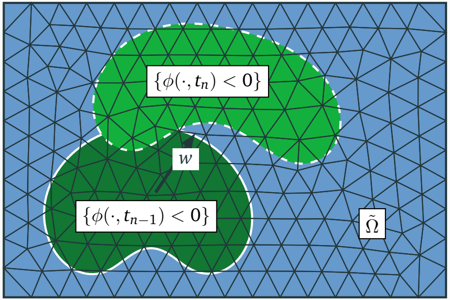
Numerical integration on (time-dependent) level set domains



Level set function:

$$\phi(x,t) \begin{cases} < 0 & x \in \Omega(t), \\ = 0 & x \in \partial\Omega(t), \\ > 0 & x \in \tilde{\Omega} \setminus \Omega(t). \end{cases}$$



Level set function:

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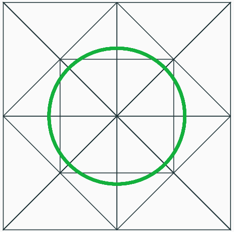
Challenge for unfitted higher order FEM on levelset domains:

How to compute integrals over

$$Q^n = \bigcup_{t \in I_n} \{(x,t) \mid \phi(\cdot, t) < 0\}$$

with higher order?

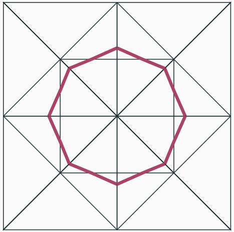
Isoparametric unfitted FEM for stationary geometries[14]



$\{\phi_h = 0\}$

implicit, higher order

+

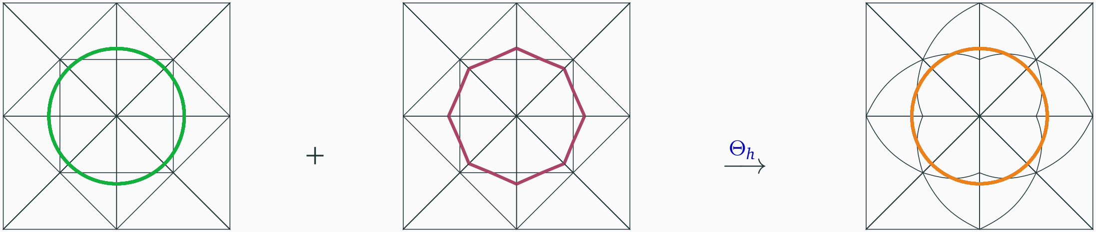


$\{\hat{\phi}_h = 0\}$

explicit, only 2^{nd} order

[14] C.L., High order unfitted finite element methods on level set domains using isoparametric mappings. CMAME, 2016

Isoparametric unfitted FEM for stationary geometries[14]



$\{\phi_h = 0\}$ implicit, higher order

$\{\hat{\phi}_h = 0\}$ explicit, only 2^{nd} order

$\Theta_h(\{\hat{\phi}_h = 0\})$ explicit, higher order

Construct parametric mapping Θ_h of underlying mesh such that $\hat{\phi}_h \approx \phi_h \circ \Theta_h$:

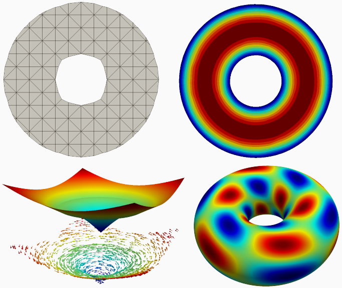
$$\rightsquigarrow \operatorname{dist}\left(\Gamma_h, \partial\left(\Theta_h(\Gamma^{\text{lin}})\right)\right) \leq \mathcal{O}(h^{k_s+1}).$$

Allows to work with $\{\hat{\phi}_h = 0\}$ as reference and guarantees robust quadrature.

[14] C.L., High order unfitted finite element methods on level set domains using isoparametric mappings. CMAME, 2016

Isoparametric unfitted FEM for stationary problems

- Scalar Interface problem[15,16] (Nitsche, 👻-penalty)
- Fictitious domain problem[17] (Nitsche, 👻-penalty)
- Stokes Interface problem[18] (..., Taylor–Hood)
- (Scalar) Surface PDEs[19] (Normal extension stabilization)



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15 C.L., A. Reusken. Analysis of a high order unfitted finite element method for an elliptic interface problem. IMA J. Numer. Anal., IMA JNA, 2018

16 C.L., A. Reusken. L2-estimates for a high order unfitted finite element method for elliptic interface problems. Journal Num. Math., 2018

17 C.L.. A higher order isoparametric fictitious domain method for level set domains, Geometrically Unfitted FEM and Appl., Springer, 2017

18 P. Lederer, C.-M. Pfeiler, C. Wintersteiger, C.L.. Higher order unfitted FEM for Stokes interface problems. PAMM, 2016

19 J. Grande, C.L., A. Reusken. Analysis of a high-order trace finite element method for PDEs on level set surfaces. SINUM, 2018

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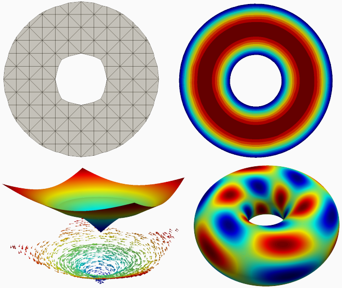

Higher order a priori error bounds for all these discretizations!

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[15] C.L., A. Reusken. Analysis of a high order unfitted finite element method for an elliptic interface problem. IMA J. Numer. Anal., IMA JNA, 2018

[16] C.L., A. Reusken. L2-estimates for a high order unfitted finite element method for elliptic interface problems. Journal Num. Math., 2018

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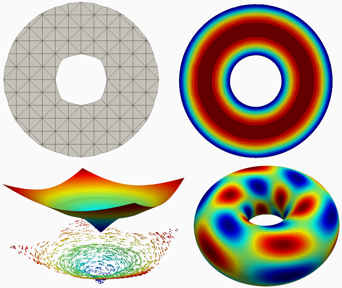

Higher order a priori error bounds for all these discretizations!

Can we design a space-time mesh transformation with the same impact?

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[15] C.L., A. Reusken. Analysis of a high order unfitted finite element method for an elliptic interface problem. IMA J. Numer. Anal., IMA JNA, 2018

[16] C.L., A. Reusken. L2-estimates for a high order unfitted finite element method for elliptic interface problems. Journal Num. Math., 2018

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[19] J. Grande, C.L., A. Reusken. Analysis of a high-order trace finite element method for PDEs on level set surfaces. SINUM, 2018

Construction of space-time mesh transformation (basic idea)

- In the space-time setting, we assume $\phi_h \in V_h^{q_s} \otimes \mathcal{P}^{q_t}(I_n)$.

[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

Construction of space-time mesh transformation (basic idea)

- In the space-time setting, we assume $\phi_h \in V_h^{q_s} \otimes \mathcal{P}^{q_t}(I_n)$.
- $\{\ell_0, \ldots, \ell_{q_t}\}$ basis of $\mathcal{P}^{q_t}(I_n) \rightsquigarrow \phi_h(x,t) = \sum_{i=0}^{q_t} \ell_i(t) \cdot \phi_h^i(x)$, for $\phi_h^i \in V_h^{q_s}$.

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- Then, the isoparametric mapping is

approximate: $\phi(x,t) \to \phi_h(x,t)$

restrict → apply $\mathcal{C}$ → sum up

$$\phi_h^0(x) \dashrightarrow \Theta_{h,0} = \mathcal{C}(\phi_h^0(x), I_h^1 \phi_h^0(x), \mathcal{T}_{h,n}^\Gamma)$$

$$\phi_h^1(x) \dashrightarrow \Theta_{h,1} = \mathcal{C}(\phi_h^1(x), I_h^1 \phi_h^1(x), \mathcal{T}_{h,n}^\Gamma)$$

$$\ldots \qquad \ldots$$

$$\phi_h^{q_t}(x) \dashrightarrow \Theta_{h,q_t} = \mathcal{C}(\phi_h^{q_t}(x), I_h^1 \phi_h^{q_t}(x), \mathcal{T}_{h,n}^\Gamma)$$

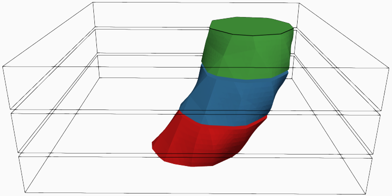
$$\Theta_h^n = \sum_{i=0}^{q_t} \ell_i(t) \Theta_{h,i}(x)$$

$$\Omega^h(t) := \Theta_h^{\mathrm{st}}(\Omega^{\mathrm{lin}}(t), t), \quad Q^{\mathrm{lin},n} = \bigcup_{t \in I_n} \Omega^{\mathrm{lin}}(t) \times \{t\}, \quad Q^{h,n} = \Theta_h^{\mathrm{st}}(Q^{\mathrm{lin},n}).$$

[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

Space-time reference configuration and isoparametric mapping

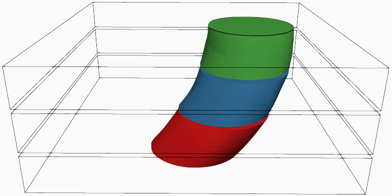
$\{\hat{\phi}_h(x,t) = 0\}, \hat{\phi}_h(x,t) \in V_h^1 \otimes \mathcal{P}^{q_t}$



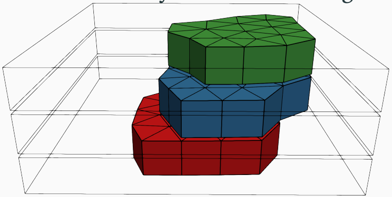
piecewise linear-in-space zero level set
(allows for arbitrary order num. integration)

$\xrightarrow{\Theta_h}$

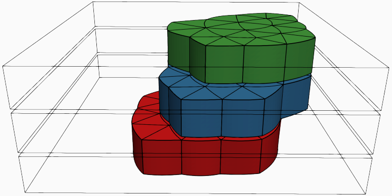
$\Theta_h(\{\hat{\phi}_h(x,t) = 0\}), \Theta_h(t) : \tilde{\Omega} \to \tilde{\Omega} \in [V_h^{q_s}]^d \otimes \mathcal{P}^{q_t}$



explicit space-time level set domain



active space-time mesh $\mathcal{E}(Q^n)$



mapped mesh

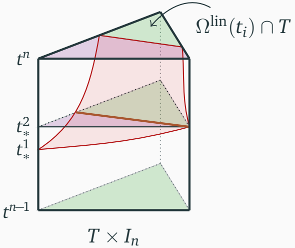
[20] J. Preuß, Higher order unfitted isoparametric space-time FEM on moving domains, Ma. thesis, Univ. Göttingen, 2018

[21] F. Heimann and C.L.. Numerical integration on hyperrectangles in isoparametric unfitted finite elements. Proc. ENUMATH 2017, 2019

Space-time quadrature on a cut reference prism

$$(f, v)_{Q^{h,n}} = \int_{Q^{h,n}} fv \, \mathrm{d}x = \int_{Q^{\text{lin},n}} |\det D\Theta_h^{\text{st}}| (f \circ \Theta_h^{\text{st}}) \hat{v} \, \mathrm{d}\hat{x}, \quad v = \hat{v} \circ \Theta_h^{\text{st}}$$

$$= \sum_{T \in \mathcal{T}_h} \int_{Q^{\text{lin},n} \cap (T \times I_n)} |\det D\Theta_h^{\text{st}}| (f \circ \Theta_h^{\text{st}}) \hat{v} \, \mathrm{d}\hat{x}$$



Space-time quadrature on a cut reference prism

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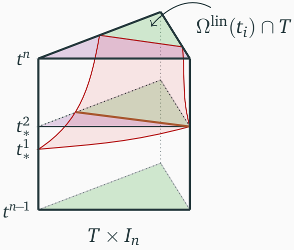
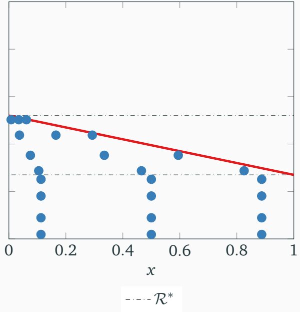


Subdivision strategy before iterated integration



1: Set $\mathcal{R} = \{t_{n-1}, t_n\}$ (time points w. cut top. changes).
2: **for** $v \in V$ (set of vertices of T) **do**
3: Search for roots $\mathcal{R}_v$ of $\phi_v : I_n \to \mathbb{R}, t \mapsto \phi^{\mathrm{lin}}(v,t)$.
4: Set $\mathcal{R} \leftarrow \mathcal{R} \cup \mathcal{R}_v$.
5: **end for**
6: Define $\mathcal{R}^*$ as set of intervals with endpoints according to $\mathcal{R}$.
7: **return** $\mathcal{R}^*$.

On isoparametric unfitted space-time FEM

Geometry error analysis[22,23]

- Higher order geometry approximation error bounds
- Shape regular deformed meshes (bounded on Θ_h^n)
- Two *blending* variants:
  - Standard blending from cut to uncut elements: can lead to small discontinuities (away from the cuts) [needs transfer operation between meshes]
  - A smooth blending allows to avoid these discontinuities

Unfitted Space-time DG error analysis with geometry error

- Higher order discretization error bounds for bulk PDEs[23]
- Higher order discretization error bounds for surface PDEs[24]

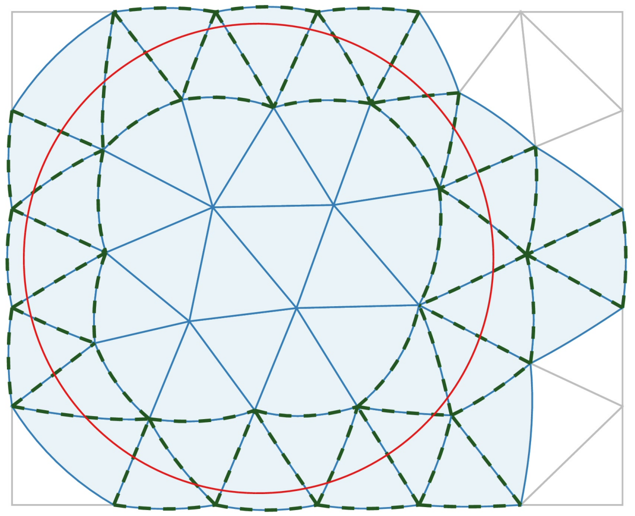
[22] F. Heimann, C.L., Geometrically higher order unfitted space-time methods for PDEs on moving domains: Geometry error analysis, arxiv:2311.02348

[23] F. Heimann, Higher Order Unfitted Space-Time Finite Element Methods for Moving Domain Problems, PhD thesis, Göttingen, 2025

[24] A. Reusken, H. Sass, Analysis of a space-time unfitted finite element method for PDEs on evolving surfaces, arXiv:2401.01215

Numerical examples

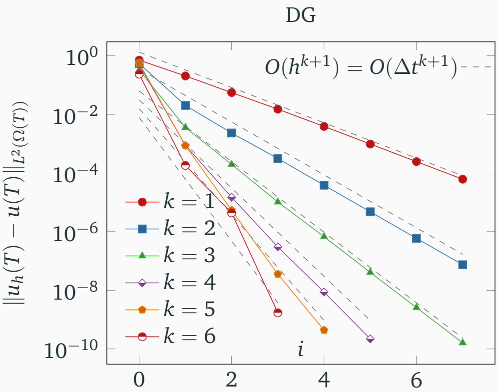
Numerical example: Moving and deforming kite

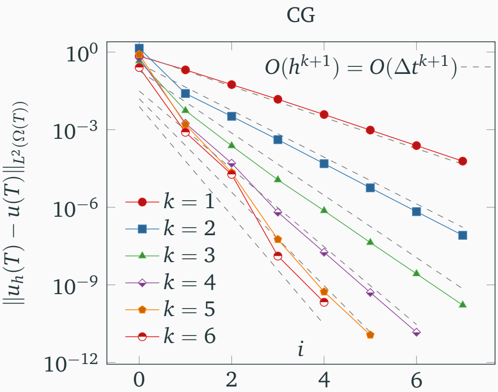


[23] F. Heimann, Higher Order Unfitted Space-Time Finite Element Methods for Moving Domain Problems, PhD thesis, Göttingen, 2025

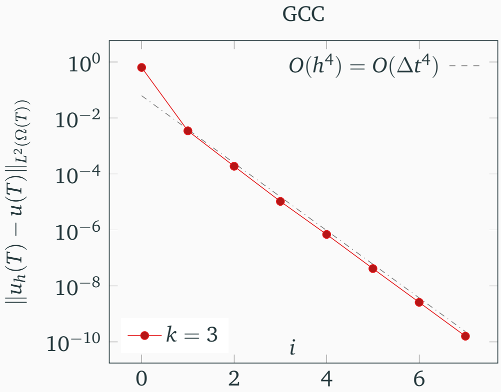
Numerical example: Moving and deforming kite

$k = k_s = k_t = q_s = q_t$, $i = i_s = i_t$, manufactured r.h.s. f.





[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023



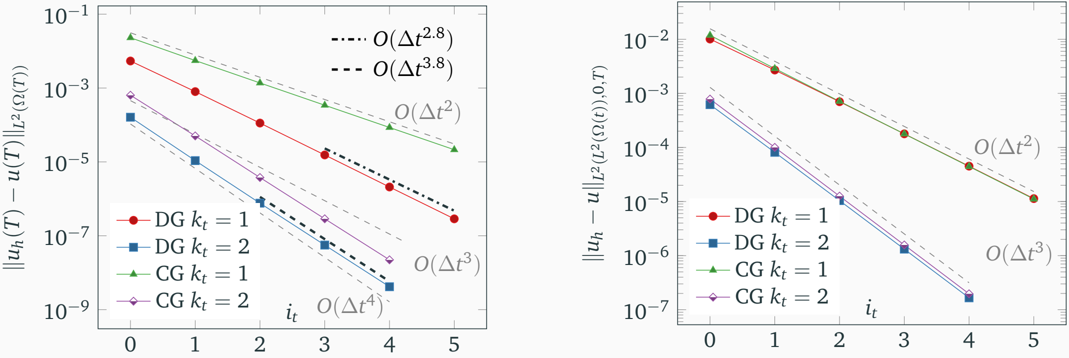
Overall, we observe

$$\|u - u_h\|_{L^2(\Omega(T))} + \|u - u_h\|_{L^2(L^2(\Omega(t)),0,T)} = \mathcal{O}(h^{k+1}) = \mathcal{O}(\Delta t^{k+1}).$$

[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

Superconvergence investigation (for the kite)

$(k_t, k_s) = (1,3), (2,5)$ and $q_t = q_s = 3, 5$

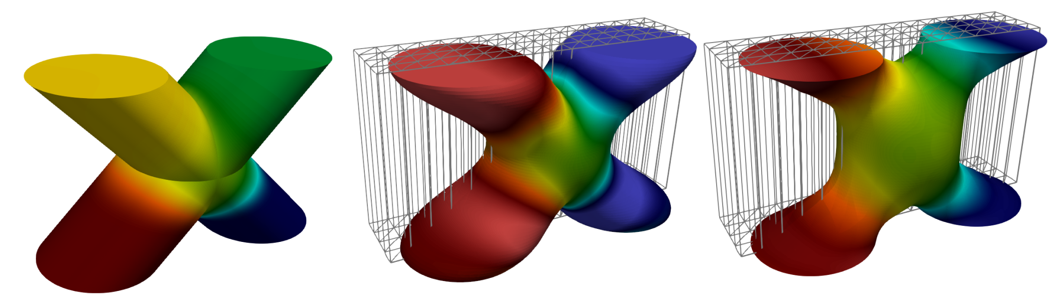


Result: $\mathcal{O}(\Delta t^{k_t+1.8})$ for DG. For CG no unique result.

[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

Colliding circles

- A test case in 2D with topology change:
- Two circles merge and seperate afterwards. Diffusion acts only in the merged setting.
- This topology change test case in one time step:



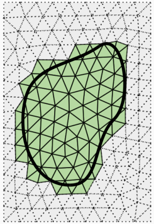
[10] F. Heimann, C.L., J. Preuß. Geometrically Higher Order Unfitted Space-Time Methods for PDEs on Moving Domains, SISC 2023

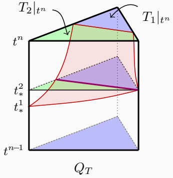
Library on top of `NGSolve`[25] for unfitted FEM: `ngsxfem`[26,27]

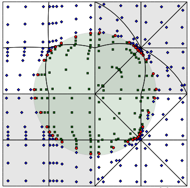
- num. integration on (multiple) level sets
- Cut FE spaces, 👻-penalties, AggFEM,
- space-time finite elements
- (2D / 3D + time) ...

```
a = BilinearForm(st_fes)
a += dt(u) * v * dQ
a += (alpha * InnerProduct(grad(u), grad(v))) * dQ
a += InnerProduct(w, grad(u)) * v * dQ
a += u * v * dOmold
a += h**(-2) * (1 + delta_t / h) * gamma * \
    (u - u.Other()) * (v - v.Other()) * dw

f = LinearForm(st_fes)
f += coeff_f * v * dQ
f += u_last * v * dOmold
```







25 `www.ngsolve.org`

26 `github.com/ngsxfem/ngsxfem`

27 C.L., F. Heimann, J. Preuß, H. v. Wahl. `ngsxfem` : Add-on to NGSolve for geometrically unfitted FEM. joss.03108 (under review), 2021.

Conclusion

Summary

- Time integration on moving domains (in Eulerian frame) is non-trivial
- Strategies for time integration and geometry handling
- Provably stable and higher order accurate
  - Eulerian time stepping
  - Space-time isoparametric unfitted FEM
- Robust realizations in 2D and 3D (+time) including geometry error analysis

Outlook

- Analysis of Petrov-Galerkin-in-time (CG, GCC, ..) variants
- Efficiency (linear solvers, preconditioners, ...)
- More complex problems (PDE / coupling to evolution / FSI)

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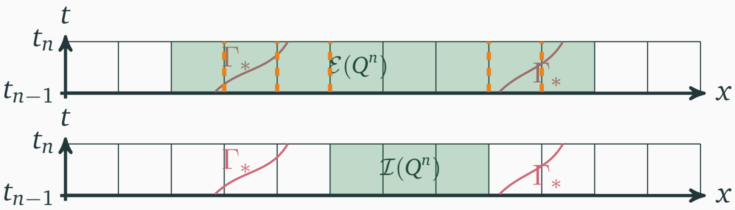
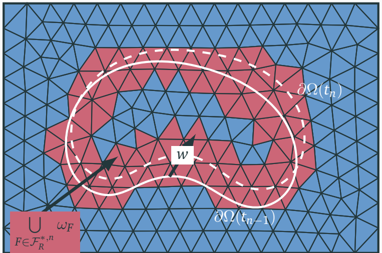
🙏 Thank you for your attention!

Back-up slides

Illustration: Minimal choice of facet patches



$$\mathcal{F}_R^{*,n} := \{F \in \mathcal{F} \ : \ F = T_1 \cap T_2, \ T_1 \in \mathcal{E}(\Omega^n) \setminus \mathcal{I}(\Omega^n), \ T_2 \in \mathcal{E}(\Omega^n)\}.$$



Assumption A.1 for analysis

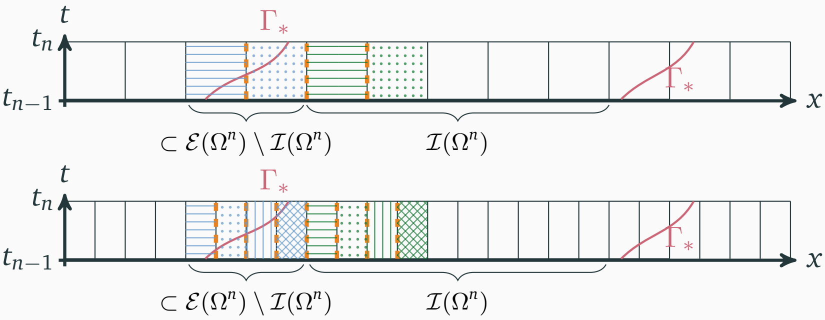
There exists a mapping (between elements) $\mathcal{B} : \mathcal{E}(\Omega^n) \to \mathcal{I}(\Omega^n)$ such that:

- The number of elements $T \in \mathcal{E}(\Omega^n)$ that map to a specific element $T_0 \in \mathcal{I}(\Omega^n)$ can be bounded independently of h and Δt, i.e. $\#(\mathcal{B}^{-1}(T_0)) \leq C$.
- For $T \in \mathcal{E}(\Omega^n) \setminus \mathcal{I}(\Omega^n)$ let $\{T_i\}_{i=0}^{M}$ be the set of elements that need to be crossed in order to traverse from $T_M = T$ to $T_0 = \mathcal{B}(T)$. Then the facets $\{T_i \cap T_j \mid i,j = 1, \ldots, M;\ i \neq j\}$ are contained in $\mathcal{F}_R^{*,n}$.
- The thickness of the layer of cut elements is bounded as

$$\#\{T \in \mathcal{E}(\Omega^n) \setminus \mathcal{I}(\Omega^n)\} \leq C_{\mathrm{B}} \left(1 + \frac{\Delta t}{h}\right)$$

  with C_{B} independent of h and Δt.

Illustration of Assumption A.1



Practical realization of A.1
Expand stabilization to small band inside domain including $\approx$ as many elements as are cut by the boundary.

Control on cut elements

Lemma
Under assumption A.1 there exists a constant $C > 0$ *such that for every* $u \in W_h \oplus \nabla W_h$ *there holds*

$$\|u\|^2_{\mathcal{E}(Q^n)} \leq C\left(\frac{h^2}{\gamma_J} j_h^n(u,u) + \|u\|^2_{\mathcal{I}(Q^n)}\right).$$

Key result for analysis

- bound norm of discrete function on $\mathcal{E}(Q^n)$ by its norm on $\mathcal{I}(Q^n)$ plus corresponding stabilization terms
- allows to extend estimates for finite elements with tensor product structure to unfitted case

A priori error estimate (no geometry error)

Theorem
Let u be the solution of the continuous problem and u_h be the discrete solution. Let $k_{\max} = \max\{k_s, k_t\}$ and assume $u \in H^{k_{\max}+2}(Q)$. Then there holds:

$$|||u - u_h||| \leq C \left(\Delta t^{k_t+1/2} + \sqrt{\left(1 + \frac{\Delta t}{h}\right) h^{k_s}} \right) \|u\|_{H^{k_{\max}+2}(Q)},$$

(semi-)norm $\lvert\cdot\rvert = \ldots$	approximation error $\inf_{w_h \in W_h} \lvert u - w_h \rvert \leq C \cdot \ldots$	discretization error $\lvert u - u_h \rvert \leq C \cdot \ldots$
$\|\partial_t \cdot\|_Q$	$\Delta t^{k_t} + h^{k_s+1}$	$\Delta t^{k_t} + \sqrt{\left(1+\frac{\Delta t}{h}\right)\frac{h^{k_s}}{\Delta t^{1/2}}}$
$\|\nabla \cdot\|_Q$	$\Delta t^{k_t+1} + h^{k_s}$	$\Delta t^{k_t+1/2} + \sqrt{\left(1+\frac{\Delta t}{h}\right) h^{k_s}}$
$\|\cdot\|_{\Omega(T)}$	$\Delta t^{k_t+1/2} + \Delta t^{-1/2} h^{k_s+1}$	$\Delta t^{k_t+1/2} + \sqrt{\left(1+\frac{\Delta t}{h}\right) h^{k_s}}$

Isoparametric FE-spaces

- integrals calculated w.r.t. piecewise planar reference configuration

$$\int_{\Theta_h(\Omega^{\text{lin}})} f \, dx = \int_{\Omega^{\text{lin}}} f \circ \Theta_h \left|\det\left(D\Theta_h\right)\right| \, dy$$
$$= \sum_{T \in \mathbb{T}_h} \sum_i \omega_i \left|\det\left(D\Theta_h(y_i)\right)\right| f(\Theta_h(y_i)).$$

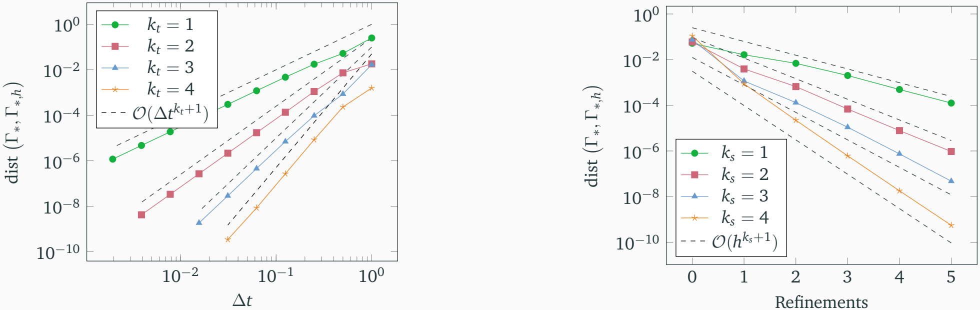
- leads to isoparametric FE spaces:

$$\mathcal{V}_h = \{v_h \circ \Theta_h^{-1} \mid v_h \in V_h\}.$$

where V_h is FE space corresponding to the piecewise planar approximation with $\hat{\phi}_h$

Test problem: circle moving through mesh

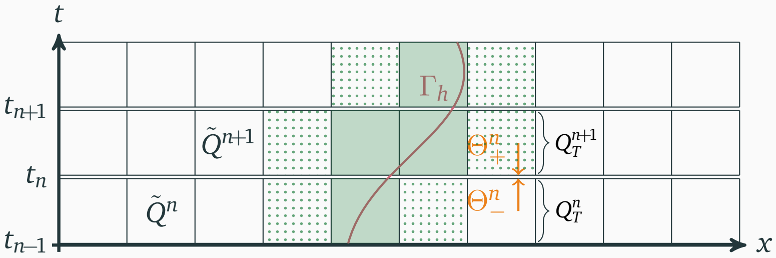
- space-time interface $\Gamma_* = \cup_{t\in(0,T]}\Gamma(t) \times \{t\}$, where $\Gamma(t) = \{\phi(\cdot,t) = 0\}$
- approximation $\Gamma_{*,h} = \cup_{t\in(0,T]}\Gamma_h(t) \times \{t\}$, where $\Gamma_h(t) = \Theta_h(t)(\Gamma^{\text{lin}}(t))$.



Observation: $\text{dist}\left(\Gamma_*, \Gamma_{*,h}\right) \lesssim \Delta t^{k_t+1} + h^{k_s+1}$

Discontinuity of mesh deformation between time slabs

Our current contruction does not ensure continuity of Θ_h between time slabs $\rightsquigarrow$ possibly $\Theta^n_+ \neq \Theta^n_-$.



Attention

Variational formulation involves passing on solution on Q^n as initial condition for variational formulation on Q^{n+1} $\rightsquigarrow$ needs treatment !

Isoparametric space-time discretization

Consider integrated by parts version of varitational formulation.

Find u such that for all v:

$$
\begin{aligned}
(u, -\partial_t v - \mathbf{w} \cdot \nabla v)_{Q^n} &+ (\nabla u, \nabla v)_{Q^n} \\
+\left(u_-^n, v_-^n\right)_{\Omega^n} &+ j_h^n(u,v) \;=\; (f, v)_{Q^n} + \left(u_-^{n-1}, v_+^{n-1}\right)_{\Omega^{n-1}}.
\end{aligned}
$$

Transition to isoparametric FEM

- take u, v from isoparametric space-time FE space $W_{n,\Theta_h} := \{v \mid v(t, \Theta_h(t,\hat{x})) = \hat{v}(t,\hat{x}) \text{ for } \hat{x} \in \Omega^{\text{lin}}(t), \text{ with } \hat{v} \in W_n\}$. Here $\hat{v} : Q^{n,\text{lin}} \mapsto \mathbb{R}$ function on undeformed mesh.
- approximate integrals by $\int\limits_{Q^n} f \approx \int\limits_{\Theta_h(Q^{n,\text{lin}})} f$, where $\Theta_h(Q^{n,\text{lin}}) = \bigcup_{t \in I_n} \Theta_h(t)\left(\Omega^{\text{lin}}(t)\right) \times \{t\}$.

Changes in variational formulation

$$(u, -\partial_t v - \mathbf{w} \cdot \nabla v)_{Q^n} + (\nabla u, \nabla v)_{Q^n} + (u_-^n, v_-^n)_{\Omega^n} + j_h^n(u, v) = (f, v)_{Q^n} + \left(u_-^{n-1}, v_+^{n-1}\right)_{\Omega^{n-1}}.$$

- contribution of 'mesh velocity':

$$\frac{d}{dt} v(t, \Theta_h(t, \hat{x})) = \frac{\partial \hat{v}}{\partial t} + \left(\frac{\partial \Theta_h}{\partial t}\right) \cdot (D\Theta_h)^{-T} \nabla \hat{v}$$

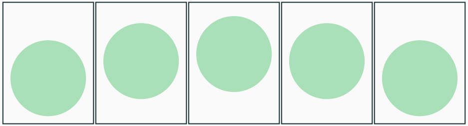
- replace u_-^{n-1} by projection Pu_-^{n-1} that fulfills

$$Pu_-^{n-1} \approx \hat{u}_-^{n-1} \circ (\Theta_+^{n-1})^{-1} = u_-^{n-1} \circ \Theta_-^{n-1} \circ (\Theta_+^{n-1})^{-1}$$

to treat possible discontinuity of Θ between time slabs.

Numerical experiment

Moving domain: circle



Manufactured solution:

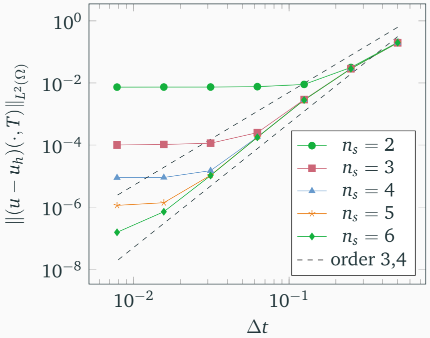
$$u(x,y,t) = \chi(\sqrt{x^2 + (y - \rho(t))^2})$$

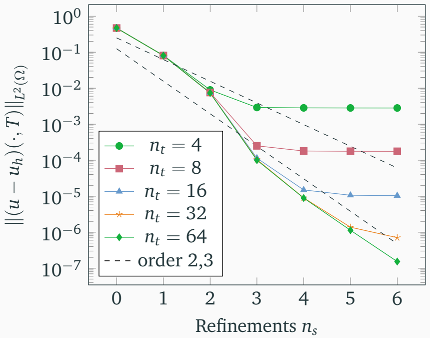
with

- $\chi(r) = \cos^2(\frac{\pi r}{2r_0}), r_0 = 1/2,$
- $\rho(t) = \frac{1}{\pi}\sin(2\pi t).$

Moving circle: L^2 norm at final time T, $k_s = 2, k_t = 2$

Notation: $\Delta t = T/n_t$, $n_s :=$ spatial refinement level.

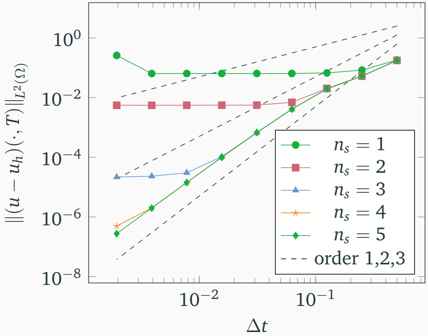




Observed rate: $\|(u - u_h)(\cdot, T)\|_{\Omega} \leq C(\Delta t^4 + h^3)$

Moving circle: Investigate superconvergence

- Choose $k_s = 3, k_t = 1$
- But third order approximation of geometry: ϕ_h, Θ_h elements of $V_h^{k_t^{\text{geom}}, k_s^{\text{geom}}}$ with $k_s^{\text{geom}} = k_t^{\text{geom}} = 3$.



Observed rate: $\|(u - u_h)(\cdot, T)\|_\Omega \leq C \Delta t^3$.

Moving circle: Summary of numerical experiments

(semi-)norm $\lvert\cdot\rvert = \dots$	approximation error $\inf_{w_h \in W_h} \lvert u - w_h \rvert \leq C \cdot \dots$	discretization error $\lvert u - u_h \rvert \leq C \cdot \dots$	numerical observation $\lvert u - u_h \rvert$
$\lVert \partial_t \cdot \rVert_Q$	$\Delta t^{k_t} + h^{k_s+1}$	$\Delta t^{k_t} + \sqrt{\left(1 + \frac{\Delta t}{h}\right) \frac{h^{k_s}}{\Delta t^{1/2}}}$	$\Delta t^{k_t} + h^{k_s+1}$
$\lVert \nabla \cdot \rVert_Q$	$\Delta t^{k_t+1} + h^{k_s}$	$\Delta t^{k_t+1/2} + \sqrt{\left(1 + \frac{\Delta t}{h}\right) h^{k_s}}$	$\Delta t^{k_t+1} + h^{k_s}$
$\lVert \cdot \rVert_Q$	$\Delta t^{k_t+1} + h^{k_s+1}$	-	$\Delta t^{k_t+1} + h^{k_s+1}$
$\lVert \nabla \cdot \rVert_{\Omega(T)}$	$\Delta t^{k_t+1/2} + \Delta t^{-1/2} h^{k_s}$	-	$\Delta t^{k_t+1} + h^{k_s}$
$\lVert \cdot \rVert_{\Omega(T)}$	$\Delta t^{k_t+1/2} + \Delta t^{-1/2} h^{k_s+1}$	$\Delta t^{k_t+1/2} + \sqrt{\left(1 + \frac{\Delta t}{h}\right) h^{k_s}}$	$\Delta t^{k_t+1+\alpha(k_t)} + h^{k_s+1}$ $\alpha(k_t) = 1$ for $k_t = 1, 2$ $\alpha(k_t = 3) > 1/2$

- numerical error converges at least as good as the bound for approximation error
- at fixed time T even converges better than approximation error estimate

Moving deforming ellipse: setup

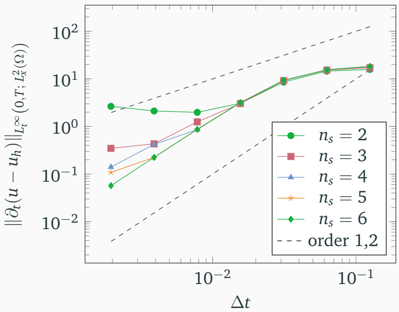
Geometry:

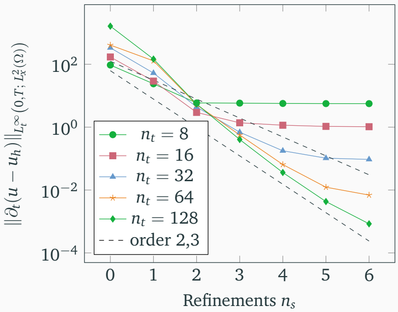
- $\phi(x,y,t) = \sqrt{[\xi(x - x_0 - \rho_x)]^2 + [\eta(y - y_0 - \rho_y)]^2} - r_0,$
- $\rho_x(t) = \frac{1}{2}\sin(4\pi t),\ \rho_y(t) = \sin(2\pi t),$
- $\xi(t) = 1 - \frac{1}{2}\sin^2(4\pi t),\ \eta(t) = 1 - \frac{1}{2}\sin^2(2\pi t)$
- $x_0 = 1, y_0 = 1/2$ and $r_0 = 1/3$.

Reference solution:

- velocity field $\mathbf{w}(t) = \left(\dot{\rho}_x(t), \dot{\rho}_y(t)\right)$
- $u(x,y,t) = \chi(\phi(x,y,t) + r_0)$ with $\chi(r) = \cos^2(\frac{\pi r}{2r_0})$
- final time $T = 1/2$

Moving deforming ellipse: convergence of time derivative





- Refinements in time with $k_s = k_t = 2$
- Refinements in space with $k_s = 2, k_t = 4$ to reduce influence of temporal error

Error analysis

Variational formulation

Summing up over time slabs ($Q = \bigcup_{n=1}^{N} Q^n$):
Find $u \in W_h$ such that for all $v \in W_h$ there holds

$$B(u,v) + J(u,v) = f(v)$$

with

$$B(u,v) := \sum_{n=1}^{N} (\partial_t u + \mathbf{w} \cdot \nabla u, v)_{Q^n} + \sum_{n=1}^{N} (\nabla u, \nabla v)_{Q^n} + \sum_{n=1}^{N-1} \left([\![u]\!]^n, v_+^n \right)_{\Omega^n} + \left(u_+^0, v_+^0 \right)_{\Omega^0},$$

$$J(u,v) := \sum_{n=1}^{N} j_h^n(u,v),$$

$$f(v) := \sum_{n=1}^{N} (f, v)_{Q^n} + \left(u_0, v_+^0 \right)_{\Omega^0}.$$

Numerical analysis (similar to Upw. DG analysis for linear hyperbol. prob.)

Main assumptions:

- exact geometry handling
- set of facets $\mathcal{F}_R^{*,n}$ for stabilization sufficiently large

Discrete norms:

$$|||u|||^2 := \sum_{n=1}^{N} \Delta t(\partial_t u, \partial_t u)_{Q^n} + [\![u]\!]^2 + (\nabla u, \nabla u)_Q, \; |||u|||_*^2 := \sum_{n=1}^{N} \left(\tfrac{1}{\Delta t} u, u\right)_{Q^n} + [\![u]\!]_*^2 + (\nabla u, \nabla u)_Q$$

with

$$[\![u]\!]^2 := \sum_{n=1}^{N-1} ([\![u]\!]^n, [\![u]\!]^n)_{\Omega^n} + \left(u_+^0, u_+^0\right)_{\Omega^0} + \left(u_-^N, u_-^N\right)_{\Omega^N}, \quad [\![u]\!]_*^2 := \sum_{n=1}^{N} \left(u_-^n, u_-^n\right)_{\Omega^n}.$$

With Ghost-penalty: $|||u|||_j^2 := |||u|||^2 + \|u\|_J^2$ and $|||u|||_{*,j}^2 := |||u|||_*^2 + \|u\|_J^2$.

Céa-like approach

- Boundedness: (crucial: ∂_t acts on W_h only)

$$B(u,v) \lesssim |||u|||_* |||u|||, \quad J(u,v) \leq \|u\|_J \|v\|, \quad \forall u \in W_h + H^1(Q),\ v \in W_h.$$

- Consistency:

$$B(u-u_h, v_h) - J(u-u_h, v_h) = 0 \quad \forall v_h \in W_h.$$

- Inf-Sup-Stability: (crucial: $\partial_t u \in W_h$)

$$\forall w_h \in W_h, \exists v_h(w_h) \in W_h,\ \text{s.t.}\ B(w_h, v_h(w_h)) + J(w_h, v_h(w_h)) \gtrsim |||w_h|||_j \cdot |||v_h(w_h)|||_j.$$

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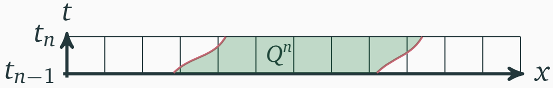
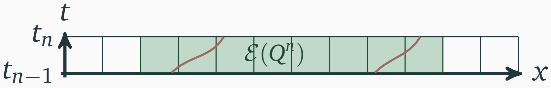
Main tool: **(restores inverse inequalities)**

For $w_h \in W_n \oplus \nabla W_n$: $\|w_h\|^2_{\mathcal{E}(Q^n)} \leq C \left(\frac{h^2}{\gamma_J} j_h^n(w_h, w_h) + \|w_h\|^2_{Q^n} \right)$.





A priori error estimate

Lemma (Céa-like result)
There holds (u: exact solution, u_h: discrete solution):

$$|||u - u_h||| \lesssim \inf_{w_h \in W_h} \left(|||u - w_h||| + |||u - w_h|||_* + \|w_h\|_J \right),$$

Theorem
There holds (u: exact solution, u_h: discrete solution): Let $k_{\max} = \max\{k_s, k_t\}$. Then there holds:

$$|||u - u_h||| \leq C \left(\Delta t^{k_t + 1/2} + \sqrt{\left(1 + \frac{\Delta t}{h}\right) h^{k_s}} \right) \|u\|_{H^{k_{\max}+2}(Q)},$$

$\rightsquigarrow$ suboptimal interpolation results (require high regularity)