Domain Decomposition Preconditioner for High Order Hybrid Discontinuous Galerkin methods

Christoph Lehrenfeld, Joachim Schöberl

DD 21

June 27th, 2012

Preliminaries

Focus of talk

- discrete system only $\rightarrow$ no error analysis
- p-version (HDG)FEM $\rightarrow$ no h-version (no geometrical coarse grid)
- tetrahedral meshes
- $\log(1) := 1$

Hybrid Discontinuous Galerkin (HDG) Method

Model problem

Approximate the solution of

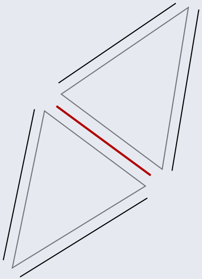
$$-\Delta u = f, \quad u = 0 \text{ on } \partial\Omega$$

on a mesh consisting of elements T and "facets" F

$$\mathcal{T} = \{T\}, \quad \mathcal{F} = \{F\}$$

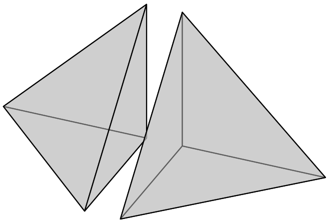
with piece-wise polynomials on elements and on facets

$$u \in P^p(\mathcal{T}), \quad \lambda \in P^p(\mathcal{F})$$



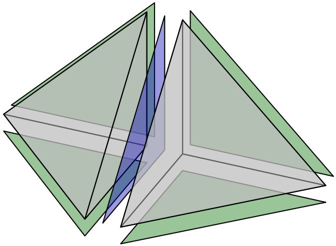
Motivation

[DG]



+ conservation
+ build in stabilization (upwinding)
+ flexibility allow for partial continuity
+ only direct neighbour communication (parallel)
+ only polynomial spaces on T

[HDG]



+ more unknowns, but less couplings, less matrix entries
+ structure allows for element-wise assembly
+ allows for static condensation of element unknowns
+ only polynomial spaces on T and F

Hybrid DG formulation for poisson

Discretization spaces

$$V_N := P^p(\mathcal{T}) \times P^p(\mathcal{F}) := \prod_{T \in \mathcal{T}} P^p(T) \times \prod_{F \in \mathcal{F}} P^p(F)$$

with according subspace $V_{N,0} = \{(u, \lambda) \in V_N : \lambda = 0 \text{ on } \partial\Omega\}$.

Hybrid DG formulation

Find $(u, \lambda) \in V_{N,0}$, s.t. $A(u, \lambda; v, \mu) = (f, v) \quad \forall\ (v, \mu) \in V_{N,0}$

with $A(u, \lambda; v, \mu) = \sum_{T \in \mathcal{T}} A_T(u, \lambda; v, \mu)$ and

$$A_T(u, \lambda; v, \mu) := \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n}(v - \mu) - \int_{\partial T} \frac{\partial v}{\partial n}(u - \lambda) + s_T(u - \lambda, v - \mu)$$

with applied manipulations for consistency, symmetry and stability.

Stabilization method - How to choose $s(\cdot,\cdot)$?

(Hybrid) Bassi Rebay

$$s_T^{HBR}(u-\lambda, v-\mu) := \sum_{F\in\mathcal{F}_T} \sigma^{HBR} \int_T r_F(u-\lambda) r_F(v-\mu), \quad \sigma^{HBR} = C$$

with $C > |\mathcal{F}_T|$ and the discrete lifting operator $r_F : P^p(F) \to [P^p(T)]^3$ defined as

$$\int_T r_F(\mu) v = \int_F \mu v \cdot n \quad \forall\ v \in [P^p(T)]^3$$

Norm to control jumps on facets: $\|\mu\|_{j,F}^2 = \|r_F(\mu)\|_{L^2(T)}^2, \quad \|\mu\|_{j,T}^2 = \sum_{F\in\mathcal{F}_T} \|\mu\|_{j,F}^2$

Stability results

Theorem

The HDG bilinear-form $A(.,.)$ is continuous and coercive on $(V_{N,0}, \|\cdot\|_{1,HDG})$ with

$$\|u\|_{1,HDG}^2 := \sum_{T\in\mathcal{T}} \|\nabla u\|_{L^2(T)}^2 + \|u-\lambda\|_{j,T}^2$$

Essential estimate for proof:

$$\|u-\lambda\|_{j,F} = \sup_{\sigma\in[P^p(T)]^3} \frac{(u-\lambda,\sigma\cdot n)_{L_2(F)}}{\|\sigma\|_{L_2(T)}}$$

$$\begin{aligned}\int_F \frac{\partial v}{\partial n}(u-\lambda) \quad &\leq \quad \|\nabla v\|_{L_2(T)} \sup_{\sigma\in[P^p]^3} \frac{\int_F \sigma_n(u-\lambda)}{\|\sigma\|_{L_2(T)}} \\ &\leq \quad \frac{1}{2\gamma}\|\nabla v\|_{L_2(T)}^2 + \frac{\gamma}{2}\|u-\lambda\|_{j,F}^2.\end{aligned}$$

$\square$

Why Bassi-Rebay jump stabilization?

Theorem

For $F \in \mathcal{F}_T$ let $\mathbf{P}^k$ denote the $L_2(F)$-orthogonal projector onto $P^k(F)$, with $\mathbf{P}^{-1} = 0$. For $\lambda \in P^p(F)$ there holds

$$\|\lambda\|_{j,F}^2 \simeq h_T^{-1} \sum_{k=0}^{p} p(p-k+1) \left\| \left(\mathbf{P}^k - \mathbf{P}^{k-1} \right) \lambda \right\|_{L_2(F)}^2$$

BR is essentially weaker than IP

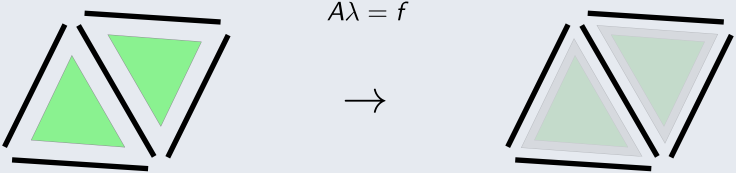
$$\frac{p}{h} \|u - \lambda\|_{L_2(F)}^2 \preceq \|u - \lambda\|_{j,F}^2 \preceq \frac{p^2}{h} \|u - \lambda\|_{L_2(F)}^2.$$

Schur complement

Eliminating element unknowns

The element unknowns can be eleminated element-wise before solving the linear system.
Consider condensated problem with the Schur complement matrix A:

$$A\lambda = f$$



facet unknowns — element unknowns — condensated

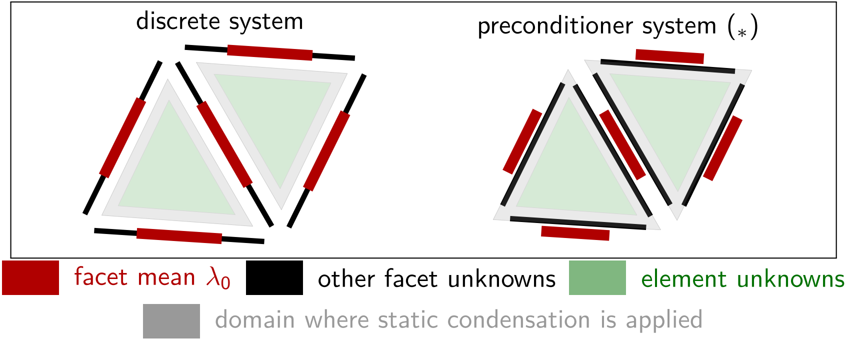
Schur complement norm

$$\|\lambda\|_A^2 := \inf_{u \in P^p(\mathcal{T})} \|(u,\lambda)\|_{1,HDG}^2$$

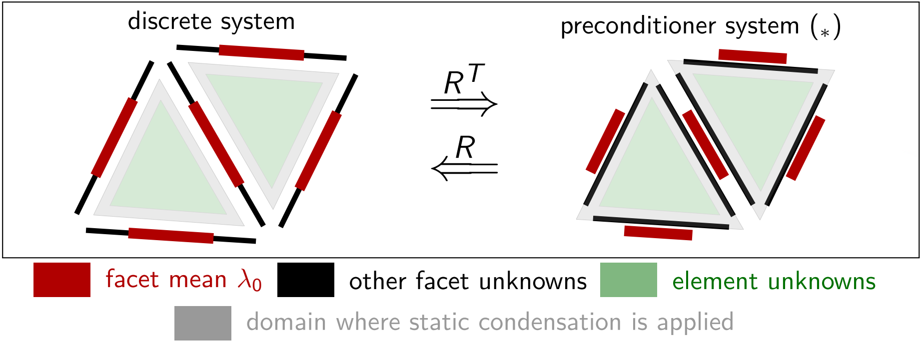
BDDC for HDG, Choice of constraints

Choice for constraint, extended system

- Decomposition: $\lambda = \lambda_0 + \lambda'$ with $\lambda_0 \in P^0(F)$ and $\lambda' \in P^0(F)^\perp$
- λ' gets doubled and becomes element-local (static condensation)
- Set only mean value on facets to be “continuous” (“constraint”)



BDDC for HDG, Communication / Averaging operator



Communication

Transfer between discrete system and the preconditioner system via:

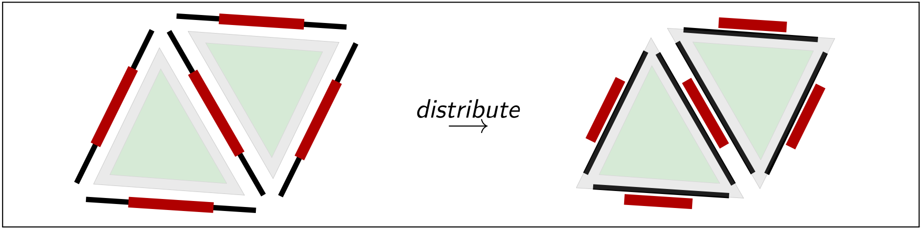
- Average (averaging operator R)
- Distribute (distribution operator R^T)

Take $R\lambda := \lambda_0 + \sum_{T:F\subset T} \lambda'_T / \sum_{T:F\subset T} 1$

BDDC algorithm

Preconditioning action $C_{BDDC}^{-1} : d \to w$, essential steps:

1. (Distribute residual) $d_* = R^T(d - Aw)$



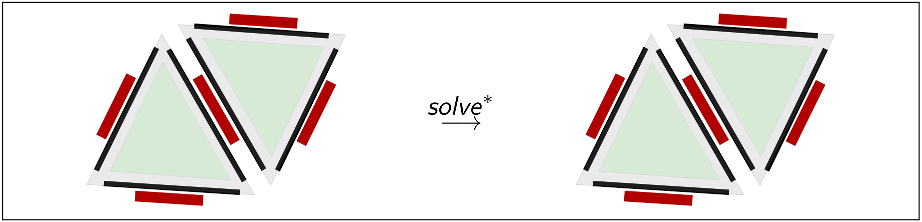
facet mean λ_0 other facet unknowns element unknowns

domain where static condensation is applied

BDDC algorithm

Preconditioning action $C_{BDDC}^{-1} : d \to w$, essential steps:

1. (Distribute residual) $d_* = R^T(d - Aw)$
2. (Solve global d_*-problem) $w_* = A_*^{-1} d_*$



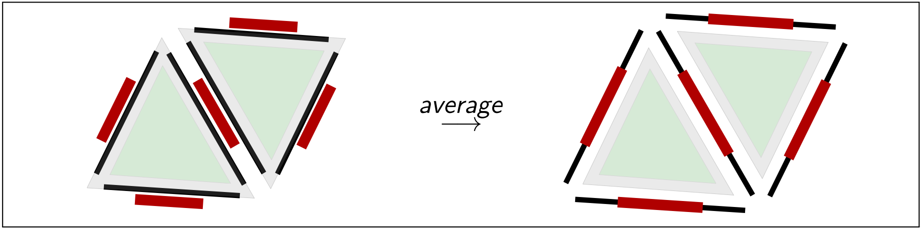
facet mean λ_0 other facet unknowns element unknowns

domain where static condensation is applied

BDDC algorithm

Preconditioning action $C_{BDDC}^{-1} : d \to w$, essential steps:

1. (Distribute residual) $d_* = R^T(d - Aw)$
2. (Solve global d_*-problem) $w_* = A_*^{-1} d_*$
3. (Average residual) $w = w + Rw_*$



facet mean λ_0 other facet unknowns element unknowns

domain where static condensation is applied

BDDC analysis / Characterizing $C_{BDDC}^{-1} = RA_*^{-1}R^T$

Fictitious space lemma

$$\|u\|_{C_{BDDC}}^2 = \inf_{\substack{y \in V_* \\ Ry = u}} \|y\|_{A_*}^2$$

and thus

$$\sigma\{C_{BDDC}^{-1}A\} \subset [1, \|R\|^2]$$

with the operator norm of $R : V_* \to V$

$$\|R\| := \sup_{\lambda \in V_*} \frac{\|R\lambda\|_A}{\|\lambda\|_{A_*}}$$

BDDC analysis / Characterizing $C_{BDDC}^{-1} = RA_*^{-1}R^T$

Fictitious space lemma

$$\|u\|_{C_{BDDC}}^2 = \inf_{\substack{y \in V_* \\ Ry = u}} \|y\|_{A_*}^2$$

and thus

$$\sigma\{C_{BDDC}^{-1}A\} \subset [1, \|R\|^2]$$

with the operator norm of $R : V_* \to V$

$$\|R\| := \sup_{\lambda \in V_*} \frac{\|R\lambda\|_A}{\|\lambda\|_{A_*}}$$

Only need estimate for $\|R\|^2$!

Bounding the averaging operator

Trace (semi-)norms $[\ \|\lambda\|_{H^{1/2}} \leftrightarrow \|\lambda\|_F^2, \quad \|\lambda\|_{H_{00}^{1/2}} \leftrightarrow \|\lambda\|_{F,0}^2]$

$$\|\lambda\|_F^2 = \inf_{u \in P^p(T)} \Big\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 \Big\}$$

$$\|\lambda\|_{F,0}^2 = \inf_{u \in P^p(T)} \Big\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F'}^2 \Big\}$$

Bounding the averaging operator

Trace (semi-)norms $[\ \|\lambda\|_{H^{1/2}} \leftrightarrow \|\lambda\|_F^2, \quad \|\lambda\|_{H_{00}^{1/2}} \leftrightarrow \|\lambda\|_{F,0}^2]$

$$\|\lambda\|_F^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 \right\}$$

$$\|\lambda\|_{F,0}^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F'}^2 \right\}$$

Bounding $R\lambda$

$$\|R\lambda\|_A^2 \quad =$$

Bounding the averaging operator

Trace (semi-)norms [$\|\lambda\|_{H^{1/2}} \leftrightarrow \|\lambda\|_F^2$, $\|\lambda\|_{H^{1/2}_{00}} \leftrightarrow \|\lambda\|_{F,0}^2$]

$$\|\lambda\|_F^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 \right\}$$

$$\|\lambda\|_{F,0}^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F'}^2 \right\}$$

Bounding $R\lambda$

$$\|R\lambda\|_A^2 = \sum_{T \in \mathcal{T}} \|R\lambda\|_{A,T}^2$$

Bounding the averaging operator

Trace (semi-)norms [$\|\lambda\|_{H^{1/2}} \leftrightarrow \|\lambda\|_F^2$, $\|\lambda\|_{H_{00}^{1/2}} \leftrightarrow \|\lambda\|_{F,0}^2$]

$$\|\lambda\|_F^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 \right\}$$

$$\|\lambda\|_{F,0}^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F'}^2 \right\}$$

Bounding $R\lambda$

$$\|R\lambda\|_A^2 = \sum_{T \in \mathcal{T}} \|R\lambda\|_{A,T}^2 \leq \sum_{T \in \mathcal{T}} \left\{ \|\lambda\|_{A,T}^2 + \|R\lambda - \lambda\|_{A,T}^2 \right\}$$

Bounding the averaging operator

Trace (semi-)norms $[\ \|\lambda\|_{H^{1/2}} \leftrightarrow \|\lambda\|_F^2, \quad \|\lambda\|_{H^{1/2}_{00}} \leftrightarrow \|\lambda\|_{F,0}^2]$

$$\|\lambda\|_F^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 \right\}$$

$$\|\lambda\|_{F,0}^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F'}^2 \right\}$$

$$\sum_{F \in \mathcal{F}_T} \|\lambda\|_{F,0}^2 \geq \|\lambda\|_{A,T}^2$$

Bounding $R\lambda$

$$\begin{aligned}
\|R\lambda\|_A^2 &= \sum_{T \in \mathcal{T}} \|R\lambda\|_{A,T}^2 \leq \sum_{T \in \mathcal{T}} \left\{ \|\lambda\|_{A,T}^2 + \|R\lambda - \lambda\|_{A,T}^2 \right\} \\
&\leq \sum_{T \in \mathcal{T}} \left\{ \|\lambda\|_{A,T}^2 + \sum_{F \in \mathcal{F}_T} \|R\lambda - \lambda\|_{F,0}^2 \right\}
\end{aligned}$$

Bounding the averaging operator

Trace (semi-)norms $\left[\ \|\lambda\|_{H^{1/2}} \leftrightarrow \|\lambda\|_F^2, \quad \|\lambda\|_{H^{1/2}_{00}} \leftrightarrow \|\lambda\|_{F,0}^2\right]$

$$\|\lambda\|_F^2 = \inf_{u\in P^p(T)}\left\{\|\nabla u\|^2_{L_2(T)} + \|u-\lambda\|^2_{j,F}\right\}$$

$$\|\lambda\|_{F,0}^2 = \inf_{u\in P^p(T)}\left\{\|\nabla u\|^2_{L_2(T)} + \|u-\lambda\|^2_{j,F} + \sum_{F'\in\mathcal{F}_T\setminus F}\|u-0\|^2_{j,F'}\right\}$$

$$\sum_{F\in\mathcal{F}_T}\|\lambda\|^2_{F,0} \geq \|\lambda\|^2_{A,T}$$

Bounding $R\lambda$

$$\begin{aligned}
\|R\lambda\|_A^2 &= \sum_{T\in\mathcal{T}}\|R\lambda\|^2_{A,T} \leq \sum_{T\in\mathcal{T}}\left\{\|\lambda\|^2_{A,T} + \|R\lambda-\lambda\|^2_{A,T}\right\} \\
&\leq \sum_{T\in\mathcal{T}}\left\{\|\lambda\|^2_{A,T} + \sum_{F\in\mathcal{F}_T}\|R\lambda-\lambda\|^2_{F,0}\right\} \\
&\preceq (\log p)^\gamma \sum_{T\in\mathcal{T}}\left\{\|\lambda\|^2_{A,T} + \sum_{F\in\mathcal{F}_T}\|R\lambda-\lambda\|^2_{F}\right\}
\end{aligned}$$

$$\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \quad \forall\,\lambda\in\{\mu\in P^p(F) : \textstyle\int_F \lambda = 0\}$$

Bounding the averaging operator

Trace (semi-)norms $[\ \|\lambda\|_{H^{1/2}} \leftrightarrow \|\lambda\|_F^2, \quad \|\lambda\|_{H_{00}^{1/2}} \leftrightarrow \|\lambda\|_{F,0}^2]$

$$\|\lambda\|_F^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 \right\}$$

$$\|\lambda\|_{F,0}^2 = \inf_{u \in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u - \lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F'}^2 \right\}$$

$$\sum_{F \in \mathcal{F}_T} \|\lambda\|_{F,0}^2 \geq \|\lambda\|_{A,T}^2$$

Bounding $R\lambda$

$$\begin{aligned}
\|R\lambda\|_A^2 &= \sum_{T \in \mathcal{T}} \|R\lambda\|_{A,T}^2 \leq \sum_{T \in \mathcal{T}} \left\{ \|\lambda\|_{A,T}^2 + \|R\lambda - \lambda\|_{A,T}^2 \right\} \leq \dots \\
&\preceq (\log p)^\gamma \sum_{T \in \mathcal{T}} \left\{ \|\lambda\|_{A,T}^2 + \sum_{F \in \mathcal{F}_T} \|R\lambda - \lambda\|_F^2 \right\} \\
&\preceq (\log p)^\gamma \sum_{T \in \mathcal{T}} \left\{ \|\lambda\|_{A,T}^2 + \sum_{F \in \mathcal{F}_T} \|\lambda\|_F^2 \right\}
\end{aligned}$$

$$\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \quad \forall\, \lambda \in \{\mu \in P^p(F) : \textstyle\int_F \lambda = 0\}$$

Bounding the averaging operator

Trace (semi-)norms $\quad [\ \|\lambda\|_{H^{1/2}} \leftrightarrow \|\lambda\|_F^2, \quad \|\lambda\|_{H^{1/2}_{00}} \leftrightarrow \|\lambda\|_{F,0}^2]$

$$\|\lambda\|_F^2 = \inf_{u\in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u-\lambda\|_{j,F}^2 \right\}$$

$$\|\lambda\|_{F,0}^2 = \inf_{u\in P^p(T)} \left\{ \|\nabla u\|_{L_2(T)}^2 + \|u-\lambda\|_{j,F}^2 + \sum_{F'\in\mathcal{F}_T\setminus F} \|u-0\|_{j,F'}^2 \right\}$$

$$\sum_{F\in\mathcal{F}_T} \|\lambda\|_F^2 \preceq \|\lambda\|_{A,T}^2$$

$$\sum_{F\in\mathcal{F}_T} \|\lambda\|_{F,0}^2 \succeq \|\lambda\|_{A,T}^2$$

Bounding $R\lambda$

$$\begin{aligned}
\|R\lambda\|_A^2 &= \sum_{T\in\mathcal{T}} \|R\lambda\|_{A,T}^2 \le \sum_{T\in\mathcal{T}} \left\{ \|\lambda\|_{A,T}^2 + \|R\lambda-\lambda\|_{A,T}^2 \right\} \preceq \ldots \\
&\preceq (\log p)^\gamma \sum_{T\in\mathcal{T}} \left\{ \|\lambda\|_{A,T}^2 + \sum_{F\in\mathcal{F}_T} \|\lambda\|_F^2 \right\} \\
&\preceq (\log p)^\gamma \sum_{T\in\mathcal{T}} \|\lambda\|_{A,T}^2 = (\log p)^\gamma \|\lambda\|_{A_*}^2
\end{aligned}$$

$$\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \quad \forall\, \lambda \in \{\mu \in P^p(F) : \textstyle\int_F \lambda = 0\}$$

Estimate for trace norm (if time)

Theorem

Let $\lambda \in P^p(F)$ *with* $\int_F \lambda = 0$. *Then there holds*

$$\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F$$

with $\gamma = 3$.

Estimate for trace norm (if time)

Theorem

Let $\lambda \in P^p(F)$ with $\int_F \lambda = 0$. Then there holds

$$\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F$$

with $\gamma = 3$.

Sketch of proof.

Consider reference element only.
Let u be the minimizer corresponding to $\|\lambda\|_F$. Construct a suitable approximation $u^* \in P^p(T)$ with $u^*|_{\partial T \setminus F} = 0$.

1. Construct a $\tilde{u} \in P^p(T)$ with zero values on ∂F.

$$u_V = u - \sum_{V \in F} \mathcal{E}_{V \to T} u(V), \quad \tilde{u} = u_V - \sum_{E \in F} \mathcal{E}_{E \to T} u_V|_E$$

2. Muñoz-Sola extension: $u^* = \mathcal{E}_{F \to T} \tilde{u}|_F, \qquad \Rightarrow u^*|_F = \tilde{u},\ u^*|_{\partial T \setminus F} = 0$
3. $$\|\nabla \tilde{u}\|^2_{L_2(T)} + \|u - \tilde{u}\|^2_{j,F} \preceq \log p \|u\|^2_{H^1(T)}$$

$\square$

BDDC results

Result of BDDC analysis

$$\sigma\{C_{BDDC}^{-1}A\} \subset [1, C(\log p)^3]$$

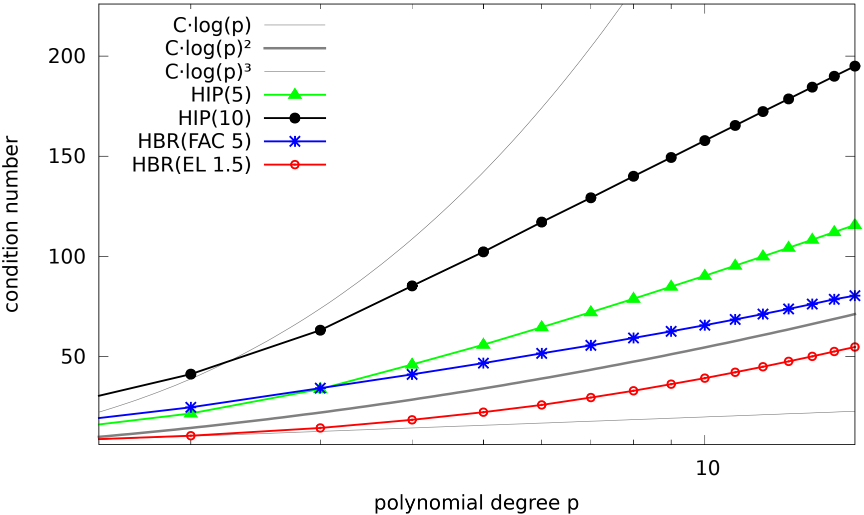
Remark

The estimate

$$\|\lambda\|_{F,0} \preceq (\log p)^{\gamma}\|\lambda\|_F \quad \forall\lambda, s.t. \int_F \lambda = 0$$

does not automatically carry over if the Hybrid Interior Penalty method defines the discrete trace norms.

3D poisson on cube with 184 els.



Theoretical result appears not to be sharp!

Conclusion

HDG methods [Cockburn,Lazarov,Gopalakrishnan 09; Cockburn et al. 07+]

! HDG methods are reasonable adaptations of DG methods when it comes to solving linear systems

! HDG methods facilitate the construction of efficient preconditioners

? Choice of (H)DG variants (HIP/HBR) is important

BDDC [Dohrmann 03; Li,Widlund 06; Pavarino, ...]

! Easy to implement (static condensation, average, distribute)

! *Natural* for HDG finite elements (static condensation^2)

! Suitable for p-version/ high order HDG finite elements

! $(\log p)^3$ condition number bound for HDG on tetrahedra meshes

Conclusion

HDG methods [Cockburn,Lazarov,Gopalakrishnan 09; Cockburn et al. 07+]

- ! HDG methods are reasonable adaptations of DG methods when it comes to solving linear systems
- ! HDG methods facilitate the construction of efficient preconditioners
- ? Choice of (H)DG variants (HIP/HBR) is important

BDDC [Dohrmann 03; Li,Widlund 06; Pavarino, ...]

- ! Easy to implement (static condensation, average, distribute)
- ! *Natural* for HDG finite elements (static condensation^2)
- ! Suitable for p-version/ high order HDG finite elements
- ! $(\log p)^3$ condition number bound for HDG on tetrahedra meshes

Thank you for your attention!

Modified versions of the BR-stabilisation I

Reduce order of facet polynomials (Superconvergence)

$$\int_T r_F'(\mu)v = \int_F \mu v \cdot n \quad \forall\ v \in [P^{p-1}(T)]^3$$

Essential estimate

$$\begin{aligned} \int_F \frac{\partial v}{\partial n}(u-\lambda) &\leq \|\nabla v\|_{L_2(T)} \sup_{\sigma \in [P^{p-1}]^3} \frac{\int_F \sigma_n (u-\lambda)}{\|\sigma\|_{L_2(T)}} \\ &\leq \frac{1}{2\gamma}\|\nabla v\|^2_{L_2(T)} + \frac{\gamma}{2}\|r_F'(u-\lambda)\|^2_{L_2(T)}. \end{aligned}$$

To use one degree less for the facets than inside the element is sufficient.

Modified versions of the BR-stabilisation II

Essential estimate (minimal facet stabilisation)

$$\int_F \frac{\partial v}{\partial n}(u-\lambda) \le \|\nabla v\|_{L_2(T)} \sup_{w \in P^p \setminus \mathbb{R}} \frac{\int_F \frac{\partial w}{\partial n}(u-\lambda)}{\|\nabla w\|_{L_2(T)}} \le \|\nabla v\|^2_{L_2(T)} \|\nabla r_F^*(u-\lambda)\|^2_{L_2(T)}$$

With lifting $r_F^*(\cdot) \in P^p \setminus \mathbb{R}$: $\int_T \nabla r_F^*(\mu) \nabla v = \int_F \frac{\partial v}{\partial n}\mu \quad \forall\, v \in P^p \setminus \mathbb{R}$

$$\begin{aligned} &A_1((u,\lambda),(v,\mu)) \\ &\quad = \sum_T \left\{ \int_T \nabla u \nabla v - \int_{\partial T} \frac{\partial u}{\partial n}[\![v]\!] - \int_{\partial T} \frac{\partial v}{\partial n}[\![u]\!] + \sigma \sum_{F \in \mathcal{F}_T} \int_T \nabla r_F^*([\![u]\!]) \nabla r_F^*([\![v]\!]) \right\} \\ &A_2((u,\lambda),(v,\mu)) = \sum_T \sum_{F \in \mathcal{F}_T} \frac{p}{h} \int_F (\mathbf{P}^{p-1})[\![u]\!](\mathbf{P}^{p-1})[\![v]\!] \end{aligned}$$

All terms for A_1 are computed anyway (lifting is cheap)!

No inverse estimate, $L^2(F)$-term only controls desired norm: $\frac{p^2}{h} \to \frac{p}{h}$!

Facet unknowns appear only in $P^{p-1}(F) \Rightarrow$ "Superconvergence"!

Modified versions of the BR-stabilisation III

Essential estimate (minimal element stabilisation)

$$\int_{\partial T} \frac{\partial v}{\partial n}(u-\lambda) \leq \|\nabla v\|_{L_2(T)} \sup_{w \in P^p \setminus \mathbb{R}} \frac{\int_{\partial T} \frac{\partial w}{\partial n}(u-\lambda)}{\|\nabla w\|_{L_2(T)}} \leq \|\nabla v\|^2_{L_2(T)} \|\nabla r^*_T(u-\lambda)\|^2_{L_2(T)}$$

With lifting $r^*_T(\cdot) \in P^p \setminus \mathbb{R}$: $\displaystyle\int_T \nabla r^*_T(\mu) \nabla v = \int_{\partial T} \frac{\partial v}{\partial n} \mu \quad \forall\, v \in P^p \setminus \mathbb{R}$

$$\begin{aligned} &A_1((u,\lambda),(v,\mu)) \\ &\quad = \sum_T \left\{ \int_T \nabla u \nabla v - \int_{\partial T} \tfrac{\partial u}{\partial n} [\![v]\!] - \int_{\partial T} \tfrac{\partial v}{\partial n} [\![u]\!] + \sigma \int_T \nabla r^*_T([\![u]\!]) \nabla r^*_T([\![v]\!]) \right\} \\ &A_2((u,\lambda),(v,\mu)) = \sum_T \int_{\partial T} \tfrac{p}{h} (\mathbf{P}^{p-1}) [\![u]\!] (\mathbf{P}^{p-1}) [\![v]\!] \end{aligned}$$

Additive Schwarz-type domain decomposition

ASM

Using ASM preconditioner with

- One (global) block for $\lambda \in P^0(\mathcal{F})$
- One (local) block for each element and λ'

gives the same $(\log p)^3$ bound for the condition number.

idea of proof.

Show that the decomposition into λ_0 and λ' is a stable decomposition.
Bound $\|\lambda'\|_S^2$ in terms of $\|\lambda\|_S^2$ using the estimates relating $\|\mu\|_F$ and $\|\mu\|_{F,0}$ for $\mu \in P^0(\mathcal{F})^\perp$. □

HDG - A priori error estimates

Energy norm estimates

From discrete stability, consistency and continuity w.r.t. to a slidely stronger norm, it follows:

$$\|(u-u_h, u-\lambda_h)\|_{1,HDG} \leq \inf_{v,\mu} \|(u-v, u-\mu)\|^*_{1,HDG} \leq Cp^\gamma \frac{h^s}{p^s} \|u\|_{H^{1+s}}$$

with $1 \leq s \leq p$ and $\gamma = 0$ on conforming meshes and $\gamma = 1/2$ else.

Full Proof of $\|\lambda\|_{F,0} \preceq (\log p)^{\gamma}\|\lambda\|_F \ \forall\lambda \in P^0(F)^{\perp}$

Full proof: $u^* = \mathcal{E}_{F\to T}\tilde{u}$, $\tilde{u}$ **constructed extension of** u **with** $\tilde{u} = 0$ **on** ∂F.

$$\|\lambda\|_{F,0}^2 \quad = \quad \inf_{u\in P^p}\left\{\|\nabla u\|_{L^2}^2 + \|u-\lambda\|_{j,F}^2 + \sum_{F'\in\mathcal{F}_T\setminus F}\|u-0\|_{j,F}^2\right\}$$

Full Proof of $\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \ \forall \lambda \in P^0(F)^\perp$

Hint: Choose u as u^*. Note that $u^* = 0$ on $\partial T \setminus F$.

Full proof: $u^* = \mathcal{E}_{F\to T}\tilde{u}$, $\tilde{u}$ **constructed extension of** u **with** $\tilde{u} = 0$ **on** ∂F.

$$
\begin{aligned}
\|\lambda\|_{F,0}^2 \quad &= \quad \inf_{u\in P^p} \left\{ \|\nabla u\|_{L^2}^2 + \|u-\lambda\|_{j,F}^2 + \sum_{F'\in\mathcal{F}_T\setminus F} \|u-0\|_{j,F}^2 \right\} \\
&\le \quad \|\nabla u^*\|^2 + \|u^*-\lambda\|_{j,F}^2 + 0
\end{aligned}
$$

Full Proof of $\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \ \forall \lambda \in P^0(F)^\perp$

Hint: $|u|_{H^1} \leq \|u\|_{H^1}$ **and** $u^* = \tilde{u}$ **on** F.

Full proof: $u^* = \mathcal{E}_{F\to T}\tilde{u}$, $\tilde{u}$ **constructed extension of** u **with** $\tilde{u} = 0$ **on** ∂F.

$$\begin{aligned}
\|\lambda\|_{F,0}^2 \quad &= \quad \inf_{u \in P^p} \left\{ \|\nabla u\|_{L^2}^2 + \|u - \lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F}^2 \right\} \\
&\leq \quad \|\nabla u^*\|^2 + \|u^* - \lambda\|_{j,F}^2 + 0 \\
&\leq \quad \|u^*\|_{H^1(T)}^2 + \|\tilde{u} - \lambda\|_{j,F}^2
\end{aligned}$$

Full Proof of $\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \ \forall \lambda \in P^0(F)^\perp$

Hint: **Muñoz Sola Ext.:** $\|u^*\|_{H^1} \preceq \|\tilde{u}\|_{H^{1/2}(\partial T)} = \|\tilde{u}\|_{H^{1/2}_{00}(F)}$, **$\Delta$-Inequality**

Full proof: $u^* = \mathcal{E}_{F\to T}\tilde{u}$, $\tilde{u}$ **constructed extension of** u **with** $\tilde{u} = 0$ **on** ∂F.

$$
\begin{aligned}
\|\lambda\|_{F,0}^2 &= \inf_{u \in P^p} \left\{ \|\nabla u\|_{L^2}^2 + \|u - \lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F}^2 \right\} \\
&\leq \|\nabla u^*\|^2 + \|u^* - \lambda\|_{j,F}^2 + 0 \\
&\leq \|u^*\|_{H^1(T)}^2 + \|\tilde{u} - \lambda\|_{j,F}^2 \\
&\preceq \|\tilde{u}\|_{H^{1/2}_{00}(F)}^2 + \|\tilde{u} - u\|_{j,F}^2 + \|u - \lambda\|_{j,F}^2
\end{aligned}
$$

Full Proof of $\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \ \forall \lambda \in P^0(F)^\perp$

Hint: **Bica (Ph.D):** $\|w\|_{H^{1/2}_{00}(F)} \preceq (\log p)^2 \|w\|_{H^{1/2}(F)} \quad \forall\, w \in P^p(F)$

Full proof: $u^* = \mathcal{E}_{F\to T}\tilde{u}$, $\tilde{u}$ **constructed extension of** u **with** $\tilde{u} = 0$ **on** ∂F**.**

$$
\begin{aligned}
\|\lambda\|_{F,0}^2 &= \inf_{u\in P^p} \left\{ \|\nabla u\|_{L^2}^2 + \|u-\lambda\|_{j,F}^2 + \sum_{F'\in\mathcal{F}_T\setminus F} \|u-0\|_{j,F}^2 \right\} \\
&\leq \|\nabla u^*\|^2 + \|u^*-\lambda\|_{j,F}^2 + 0 \\
&\leq \|u^*\|_{H^1(T)}^2 + \|\tilde{u}-\lambda\|_{j,F}^2 \\
&\preceq \|\tilde{u}\|_{H^{1/2}_{00}(F)}^2 + \|\tilde{u}-u\|_{j,F}^2 + \|u-\lambda\|_{j,F}^2 \\
&\preceq (\log p)^2 \|\tilde{u}\|_{H^{1/2}(F)}^2 + \|\tilde{u}-u\|_{j,F}^2 + \|u-\lambda\|_{j,F}^2
\end{aligned}
$$

Full Proof of $\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \ \forall \lambda \in P^0(F)^\perp$

Hint: $\|v\|_{H^{1/2}(F)} \leq \|\nabla v\|_{L^2(T)}$

Full proof: $u^* = \mathcal{E}_{F\to T}\tilde{u}$, $\tilde{u}$ **constructed extension of** u **with** $\tilde{u} = 0$ **on** ∂F.

$$\begin{aligned}
\|\lambda\|_{F,0}^2 &= \inf_{u \in P^p} \left\{ \|\nabla u\|_{L^2}^2 + \|u-\lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F}^2 \right\} \\
&\leq \|\nabla u^*\|^2 + \|u^* - \lambda\|_{j,F}^2 + 0 \\
&\leq \|u^*\|_{H^1(T)}^2 + \|\tilde{u} - \lambda\|_{j,F}^2 \\
&\preceq \|\tilde{u}\|_{H_{00}^{1/2}(F)}^2 + \|\tilde{u} - u\|_{j,F}^2 + \|u - \lambda\|_{j,F}^2 \\
&\preceq (\log p)^2 \|\tilde{u}\|_{H^{1/2}(F)}^2 + \|\tilde{u} - u\|_{j,F}^2 + \|u - \lambda\|_{j,F}^2 \\
&\leq (\log p)^2 \|\nabla \tilde{u}\|_{L^2(T)}^2 + \|\tilde{u} - u\|_{j,F}^2 + \|u - \lambda\|_{j,F}^2
\end{aligned}$$

Full Proof of $\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \ \forall \lambda \in P^0(F)^\perp$

Hint: $\|\nabla \tilde{u}\|^2_{L^2(T)} + \|\tilde{u} - u\|^2_{j,F} \leq C \log p \|u\|^2_{H^1(T)}$ **(Thm. 11 + 18)**

Full proof: $u^* = \mathcal{E}_{F\to T}\tilde{u}$, $\tilde{u}$ **constructed extension of** u **with** $\tilde{u} = 0$ **on** ∂F.

$$
\begin{aligned}
\|\lambda\|^2_{F,0} &= \inf_{u \in P^p} \left\{ \|\nabla u\|^2_{L^2} + \|u - \lambda\|^2_{j,F} + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|^2_{j,F} \right\} \\
&\leq \|\nabla u^*\|^2 + \|u^* - \lambda\|^2_{j,F} + 0 \\
&\leq \|u^*\|^2_{H^1(T)} + \|\tilde{u} - \lambda\|^2_{j,F} \\
&\preceq \|\tilde{u}\|^2_{H^{1/2}_{00}(F)} + \|\tilde{u} - u\|^2_{j,F} + \|u - \lambda\|^2_{j,F} \\
&\preceq (\log p)^2 \|\tilde{u}\|^2_{H^{1/2}(F)} + \|\tilde{u} - u\|^2_{j,F} + \|u - \lambda\|^2_{j,F} \\
&\leq (\log p)^2 \|\nabla \tilde{u}\|^2_{L^2(T)} + \|\tilde{u} - u\|^2_{j,F} + \|u - \lambda\|^2_{j,F} \\
&\preceq (\log p)^3 \|u\|^2_{H^1(T)} + \|u - \lambda\|^2_{j,F}
\end{aligned}
$$

Full Proof of $\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \ \forall \lambda \in P^0(F)^\perp$

Hint: **Poincare-type inequality:** $\|u\|_{H^1(T)} \preceq \|\nabla u\|^2_{L^2(T)} + \|u-\lambda\|^2_{j,F}$.

Full proof: $u^* = \mathcal{E}_{F\to T}\tilde{u}$, $\tilde{u}$ **constructed extension of** u **with** $\tilde{u}=0$ **on** ∂F.

$$
\begin{aligned}
\|\lambda\|^2_{F,0} &= \inf_{u\in P^p}\left\{\|\nabla u\|^2_{L^2} + \|u-\lambda\|^2_{j,F} + \sum_{F'\in\mathcal{F}_T\setminus F}\|u-0\|^2_{j,F}\right\} \\
&\leq \|\nabla u^*\|^2 + \|u^*-\lambda\|^2_{j,F} + 0 \\
&\leq \|u^*\|^2_{H^1(T)} + \|\tilde{u}-\lambda\|^2_{j,F} \\
&\preceq \|\tilde{u}\|^2_{H^{1/2}_{00}(F)} + \|\tilde{u}-u\|^2_{j,F} + \|u-\lambda\|^2_{j,F} \\
&\preceq (\log p)^2\|\tilde{u}\|^2_{H^{1/2}(F)} + \|\tilde{u}-u\|^2_{j,F} + \|u-\lambda\|^2_{j,F} \\
&\leq (\log p)^2\|\nabla\tilde{u}\|^2_{L^2(T)} + \|\tilde{u}-u\|^2_{j,F} + \|u-\lambda\|^2_{j,F} \\
&\preceq (\log p)^3\|u\|^2_{H^1(T)} + \|u-\lambda\|^2_{j,F} \\
&\preceq (\log p)^3(\|\nabla u\|^2_{L^2(T)} + \|u-\lambda\|^2_{j,F})
\end{aligned}
$$

Full Proof of $\|\lambda\|_{F,0} \preceq (\log p)^\gamma \|\lambda\|_F \ \forall \lambda \in P^0(F)^\perp$

Hint: u **is the minimizer corresponding to** $\|\lambda\|_F$**.**

Full proof: $u^* = \mathcal{E}_{F\to T}\tilde{u}$**,** $\tilde{u}$ **constructed extension of** u **with** $\tilde{u} = 0$ **on** ∂F**.**

$$\begin{aligned}
\|\lambda\|_{F,0}^2 &= \inf_{u \in P^p} \left\{ \|\nabla u\|_{L^2}^2 + \|u - \lambda\|_{j,F}^2 + \sum_{F' \in \mathcal{F}_T \setminus F} \|u - 0\|_{j,F}^2 \right\} \\
&\leq \|\nabla u^*\|^2 + \|u^* - \lambda\|_{j,F}^2 + 0 \\
&\leq \|u^*\|_{H^1(T)}^2 + \|\tilde{u} - \lambda\|_{j,F}^2 \\
&\preceq \|\tilde{u}\|_{H_{00}^{1/2}(F)}^2 + \|\tilde{u} - u\|_{j,F}^2 + \|u - \lambda\|_{j,F}^2 \\
&\preceq (\log p)^2 \|\tilde{u}\|_{H^{1/2}(F)}^2 + \|\tilde{u} - u\|_{j,F}^2 + \|u - \lambda\|_{j,F}^2 \\
&\leq (\log p)^2 \|\nabla \tilde{u}\|_{L^2(T)}^2 + \|\tilde{u} - u\|_{j,F}^2 + \|u - \lambda\|_{j,F}^2 \\
&\preceq (\log p)^3 \|u\|_{H^1(T)}^2 + \|u - \lambda\|_{j,F}^2 \\
&\preceq (\log p)^3 (\|\nabla u\|_{L^2(T)}^2 + \|u - \lambda\|_{j,F}^2) \\
&= (\log p)^3 \|\lambda\|_F^2
\end{aligned}$$