GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Efficient divergence-conforming finite element methods for incompressible flows

On the benefits of exact incompressibility, hybridization and operator splitting

Christoph Lehrenfeld

joint work with:

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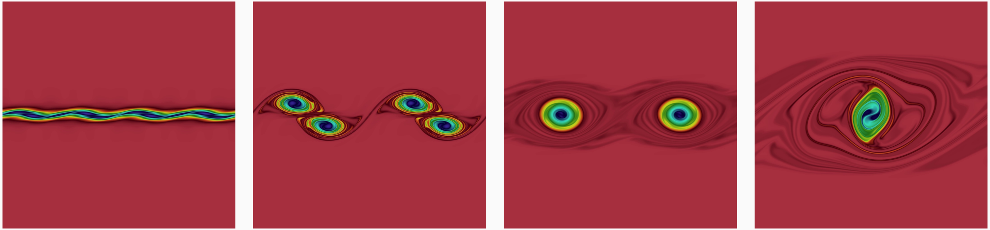
G. Fu (Univ. of Notre Dame)

IfAM, Hannover, January 30th 2020

Problem of interest: incompressible Navier-Stokes equations

$$\left\{ \begin{aligned} \frac{\partial}{\partial t} u + \operatorname{div}(-\nu \nabla u + u \otimes u) + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega. \end{aligned} \right. \quad + \text{ initial / boundary cond.}$$



[1] P.Schroeder, V.John, P.Lederer, C.L., G.Lube, J.Schöberl, On Reference Solutions [...] of the 2d KH Instab., CAMWA, 2019

Incompressible Navier-Stokes equations

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Challenges:

- stability 1: stable velocity-pressure space (a.k.a. LBB-stability)

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Challenges:

- stability 1: stable velocity-pressure space (a.k.a. LBB-stability)
- stability 2: convection domination

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- stability 3: non-linearity (energy-stability)

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- stability 1: stable velocity-pressure space (a.k.a. LBB-stability)
- stability 2: convection domination
- stability 3: non-linearity (energy-stability)
- high accuracy and efficiency

[1] P.Schroeder, V.John, P.Lederer, C.L., G.Lube, J.Schöberl, On Reference Solutions [...] of the 2d KH Instab., CAMWA, 2019

Structure of the talk

Generic FEM formulation for the Navier–Stokes equations

Find $u_h \in \Sigma_h$ and $p_h \in Q_h$ approximating $u \in [H^1]^d$, $p \in L^2$, s.t.

$$\begin{aligned} (\tfrac{\partial}{\partial t} u_h, v_h) + a_h(u_h, v_h) + c_h(u_h; u_h, v_h) + b_h(v_h, p_h) &= f(v_h) \ \forall v_h \in \Sigma_h, \\ b_h(u_h, q_h) &= \quad 0 \quad \forall q_h \in Q_h. \end{aligned}$$

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Part I: How to choose Σ_h / Q_h $(b_h(\cdot,\cdot))$? $\rightsquigarrow$ divergence/non-conforming FEM

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Part IV: Applications

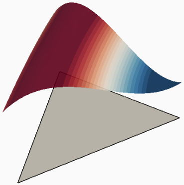
H(div)-conforming Finite Elements

The construction of finite elements

The conforming way

- Pick a polynomial space on each element

Polynomial on one element

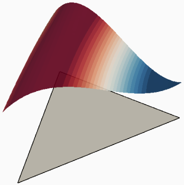


The construction of finite elements

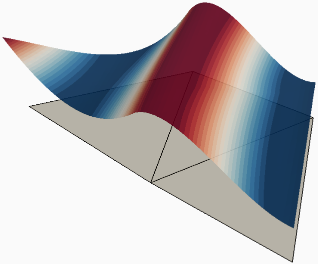
The conforming way

- Pick a polynomial space on each element
- Add constraints at element interfaces

Polynomial on one element



Here: continuity



Different conforming FE spaces

Different conditions at interface for conformity

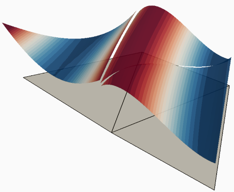
$$u|_T \in \mathcal{P}^k(T) \qquad\qquad\qquad \Rightarrow \qquad u \in L^2(\Omega)$$

$$u|_T \in \mathcal{P}^k(T) \quad \text{and} \quad [\![u]\!]_F = 0 \quad \Rightarrow \quad u \in H^1(\Omega)$$

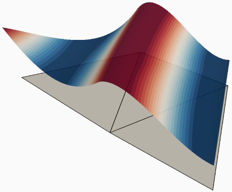
$$u|_T \in [\mathcal{P}^k(T)]^d \quad \text{and} \quad [\![u \cdot n]\!]_F = 0 \quad \Rightarrow \quad u \in H(div, \Omega)$$

$$u|_T \in [\mathcal{P}^k(T)]^d \quad \text{and} \quad [\![u \times n]\!]_F = 0 \quad \Rightarrow \quad u \in H(curl, \Omega)$$

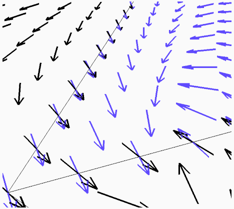
L^2**-conf.**



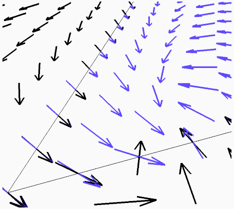
H^1**-conf.**



$H(div)$**-conf.**



$H(curl)$**-conf.**



Non-conforming FE spaces

The non-conforming way

- Pick a polynomial space on each element
- Impose (some) conditions across element interfaces only weakly

Reasons to abandon conforming setting

- (local) conservation properties
- stability for saddle-point problems
- stability for non-symmetric/non-linear problems (e.g. convection)
- simplicity of data structures / construction
- ...

Saddle point problems and compatible spaces

$$\begin{aligned} ...(u)... + \nabla p &= f && \text{in } \Omega, \\ div\, u &= 0 && \text{in } \Omega. \end{aligned}$$

Saddle point problem: How to choose velocity and pressure spaces (Σ_h / Q_h)?

$$\begin{cases} \dots \quad + b_h(v,p) = \langle f, v \rangle & \forall\, v \in \Sigma_h, \\ \qquad b_h(u,q) = 0 & \forall\, q \in Q_h, \end{cases} \qquad b_h(u,q) := -\int_\Omega div(u) q \, dx.$$

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Two effects that need to be balanced:

Stability:

pressure space not too large,

e.g. if $dim(Q_h) > dim(\Sigma_h)$

$\rightsquigarrow$ velocity overconstraint,

pressure underdetermined

Accuracy:

pressure space not too small,

$$\int_\Omega div(u) q = 0 \quad \forall q \in Q_h$$

small pressure space

$\rightsquigarrow$ poor approximation of $div(u) = 0$

Saddle point problems and compatible spaces

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Ideal case: $div(\Sigma_h) = Q_h$.

Discretization with (only) $H(div)$-conforming FEM

Abandon H^1-conformity($[\![u]\!]_F = 0$) $\rightarrow$ consider $H(div)$-conformity ($[\![u \cdot n]\!]_F = 0$)

- Velocity space BDM^k (on simplices):

$$\Sigma_h := \{u|_T \in [\mathcal{P}^k(T)]^d \text{ and } [\![u \cdot n]\!] = 0\} \subset H(div, \Omega)$$

- Pressure space:

$$Q_h := \{p|_T \in [\mathcal{P}^{k-1}(T)]\} \subset L^2(\Omega)$$

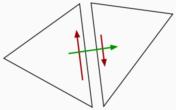
$div(\Sigma_h) = Q_h \rightsquigarrow$ **exactly divergence-free solutions**

$$u \in \Sigma_h, \quad \int_\Omega div(u)q \; dx = 0 \; \forall q \in Q_h \quad \overset{q=div(u)}{\Longrightarrow} \quad div(u) = 0 \text{ in } \Omega.$$

The benefits of $H(\mathrm{div})$-conforming FE spaces I / IV

- **local conservation property[2]: perfect mass balance**

$\mathrm{div}(u) = 0$ pointwise and normal-continuity



- **LBB-stability**

$$\inf_{p \in Q_h} \sup_{u \in \Sigma_h} \frac{\int_\Omega \mathrm{div}(u) p dx}{\|p\|_0 \|u\|_{1,h}} \geq c \neq c(h)^3, \quad \neq c(h,k)^4 \qquad (h \text{ and } k\text{-robust})$$

[2] B. Cockburn, G. Kanschat, D. Schötzau: A note on DG divergence-free solutions of the Nav.-Stokes eqs., J. Sci. Comput., 2007

[3] C.L., Hybrid DG methods for solving incompr. flow problems, Diploma thesis, RWTH Aachen, 2010

[4] P. Lederer, J. Schöberl, Polynomial robust stability analysis for H(div)-conforming [FE] for the Stokes equations, IMAJNA, 2017

The benefits of $H(\mathrm{div})$-conforming FE spaces II / IV

- **pressure robustness[5] (discussion here only for Stokes)**

With LBB-stability and assuming (for now) continuity & coercivity of $a_h(\cdot,\cdot)$:

$$\nu \|u - u_h\|_{\Sigma_h} + \|p - p_h\|_Q \leq (\nu \|u\|_{H^{k+1}} + \underbrace{\|p\|_{H^k}}_{\text{"good"?}}) c h^k$$

[5] A. Linke, On the role of the Helmholtz decomposition in [...] incompressible flows and a new variational crime, CMAME, 2014

The benefits of $H(\mathrm{div})$-conforming FE spaces II / IV

- **pressure robustness**[5] **(discussion here only for Stokes)**

Helmholtz decomposition: $f = \underbrace{g}_{\text{div.free}} + \underbrace{\nabla \phi}_{\text{rot.free}}$

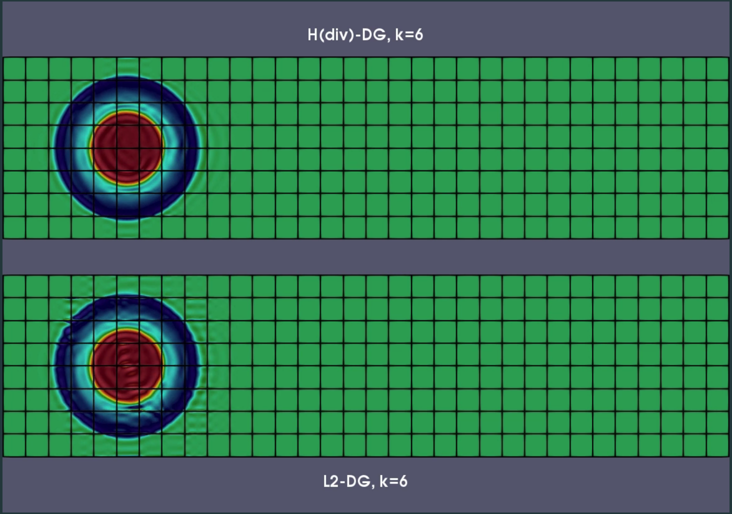
$$\begin{aligned} a_h(u_h, v_h) + b_h(v_h, p_h) &= (g + \nabla \phi, v_h) \; \forall v_h \in \Sigma_h, \\ b_h(u_h, q_h) &= \qquad 0 \qquad \forall q_h \in Q_h. \end{aligned}$$

For $u_h \in \Sigma_h^0$, the div.-free subsp.: $a_h(u_h, v_h^0) \underbrace{+ b_h(v_h^0, p_h)}_{=0} = (g + \underbrace{\nabla \phi, v_h^0}_{=0}) \quad \forall v_h^0 \in \Sigma_h^0$

u and u_h only depend on g, ϕ is balanced by p_h. $\rightsquigarrow$ $\|u - u_h\|_{\Sigma_h} = F(u)$

[5] A. Linke, On the role of the Helmholtz decomposition in [...] incompressible flows and a new variational crime, CMAME, 2014

A simple example: $H(\mathrm{div})$-conforming vs. DG



courtesy of Philipp W. Schroeder

- **Non-negativeness of convection trilinear form** $\rightsquigarrow$ **energy-stability**

$c_h(w; u, u) \geq 0$ if w div.free in $H(\text{div})$ and c_h based on a central/upwind flux.

Energy balance for $f = 0$ ($v = u, q = -p$):

$$\partial_t \frac{1}{2}(u, u) + \underbrace{a_h(u, u)}_{\geq 0} + \underbrace{c_h(u; u, u)}_{\geq 0} + \underbrace{b_h(u, p) + b_h(u, -p)}_{=0} = 0 \quad \Longrightarrow \partial_t \frac{1}{2}(u, u) \leq 0$$

- **Conservation of kin. energy, linear/angular momentum**[6]

[6] S.Charnyi, T.Heister, M.A.Olshanskii, L. Rebholz, On conservation laws of Nav.-Sto. Galerkin disc., J. Comp. Phys., 2017

The benefits of $H(\text{div})$-conforming FE spaces IV / IV

- **Reducable velocity and pressure spaces (inner bubbles)**[3]

Based on a finite element space decomposition[7]:

low order | facet bubbles (div.free) | cell bubbles (div.free) | cell bubbles (non-div.free)

- **...** $\rightsquigarrow$ **lots of benefits**

[7] J. Schöberl, S. Zaglmayr, High order Nédélec elements with local complete sequence properties, COMPEL, 2005

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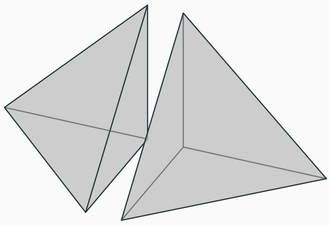
Price to pay: handle tangential continuity through var. formulation

[7] J. Schöberl, S. Zaglmayr, High order Nédélec elements with local complete sequence properties, COMPEL, 2005

DG/HDG for (tangential) continuity

(Primal) Discontinuous Galerkin in a nutshell $[\![\cdot]\!]$: jump, $\{\!\{\cdot\}\!\}$: average



- V_h: Piecewise polyn. / discontinuous
- Element unknowns couple with all neighbor unknowns

DG for scalar Poisson[8]**:**

$$-\Delta u = f \text{ in } \Omega$$

Symmetric interior penalty formulation: Find $u \in V_h$, s.t.

$$\sum_T \int_T \nabla u \nabla v \, dx + \sum_F \int_F \{\!\{-\frac{\partial u}{\partial n}\}\!\} [\![v]\!] \, ds$$

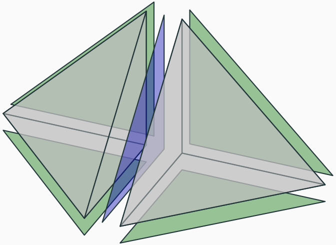
$$+ \sum_F \int_F \{\!\{-\frac{\partial v}{\partial n}\}\!\} [\![u]\!] \, ds + \sum_F \int_F \frac{\lambda}{h} [\![u]\!] [\![v]\!] \, ds = \langle f, v \rangle \; \forall \; v \in V_h$$

Many couplings

$\rightsquigarrow$ **solving linear systems is expensive!**

[8] D. Arnold, F. Brezzi, B. Cockburn, LD. Marini, Unified Analysis of DG ... Elliptic Probl., SIAM J.Num. Anal., 2002

(Primal) Hybrid Discontinuous Galerkin $[\![v]\!] = v - v_F$: facet jump (one-sided)



- V_h: Piecewise polyn. / discontinuous
- $+ F_h$: polyn. on facets
- V_h: no coupl. with neighb.
- static condensation

HDG for Poisson:

$$-\Delta u = f \text{ in } \Omega$$

Symmetric interior penalty HDG formulation: Find $(u, u_F) \in V_h \times F_h$, s.t.

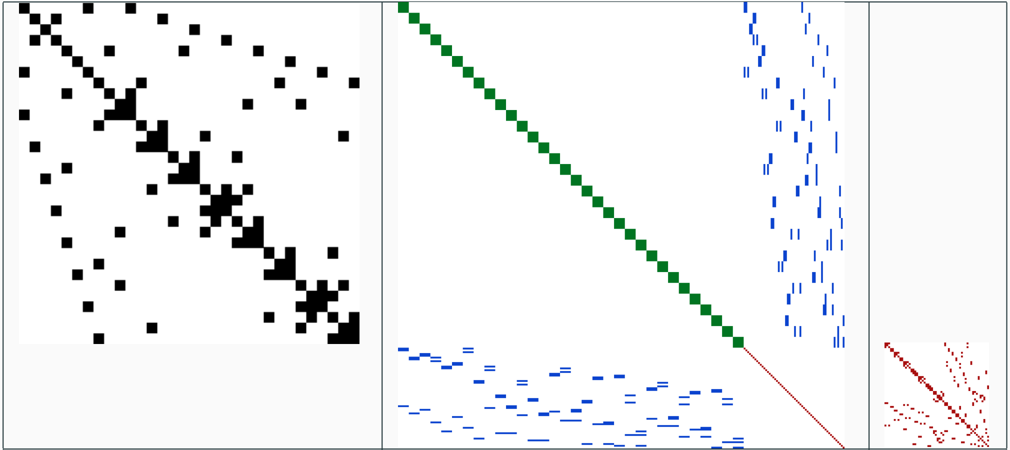
$$\sum_T \int_T \nabla u \nabla v \, dx + \int_{\partial T} -\frac{\partial u}{\partial n} [\![v]\!] \, ds$$
$$+ \int_{\partial T} -\frac{\partial v}{\partial n} [\![u]\!] \, ds + \int_{\partial T} \frac{\lambda}{h} [\![u]\!][\![v]\!] \, ds = \langle f, v \rangle, \ \forall \ (v, v_F) \in V_h \times F_h$$

More unknowns, but better sparsity

$\rightsquigarrow$ **less couplings, allows for static condensation!**

[9] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified Hybridization of DG, ... , SIAM J.Num. Anal., 2009

Example sparsity patterns (2D, $k = 10$)



DG	HDG	HDG (cond.)

Local postprocessing in mixed methods

Improving hybrid mixed solutions[9,10]

In hybrid mixed (and Hybrid (mixed) DG) discretizations one has

$$\|u_{\text{exact}} - u\|_{L^2} \lesssim h^{k+1}$$

with order k on the facets. With local postprocessing one can obtain $u^* \in V_h^{k+1,\text{disc}}$ by solving element-local problems (with weakly imposed Dirichlet-Data u_F):

$$\|u_{\text{exact}} - u^*\|_{L^2} \lesssim h^{k+2} \qquad \text{“HDG superconvergence”}$$

[9] F. Brezzi & M. Fortin, Mixed and Hybrid Finite Element Methods, 1991, Springer

[10] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified Hybridization of DG, ... , SIAM J.Num. Anal., 2009

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Can we achieve the same in a primal HDG formulation?

[9] F. Brezzi & M. Fortin, Mixed and Hybrid Finite Element Methods, 1991, Springer

[10] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified Hybridization of DG, ... , SIAM J.Num. Anal., 2009

Projected jumps

Improving primal HDG formulations[11]

$$\sum_T \int_T \nabla u \nabla v \, dx + \int_{\partial T} - \underbrace{\frac{\partial u}{\partial n}}_{\in \mathcal{P}^{k-1}(F)} [\![v]\!] \, ds + \int_{\partial T} - \underbrace{\frac{\partial v}{\partial n}}_{\in \mathcal{P}^{k-1}(F)} [\![u]\!] \, ds + \int_{\partial T} \frac{\lambda}{h} [\![u]\!] [\![v]\!] \, ds = ..$$

[11] C.L. & J. Schöberl, High order exactly divergence-free HDG Methods for incompressible... flows, 2016, CMAME

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$$\sum_T \int_T \nabla u \nabla v \, dx + \int_{\partial T} - \underbrace{\frac{\partial u}{\partial n}}_{\in \mathcal{P}^{k-1}(F)} \Pi [\![v]\!] \, ds + \int_{\partial T} - \underbrace{\frac{\partial v}{\partial n}}_{\in \mathcal{P}^{k-1}(F)} \Pi [\![u]\!] \, ds + \int_{\partial T} \frac{\lambda}{h} \Pi [\![u]\!] \Pi [\![v]\!] \, ds = ..$$

with $\Pi : L^2(F) \to \mathcal{P}^{k-1}(F)$ the facet-wise L^2 projection into $\mathcal{P}^{k-1}(F)$.

$\Longrightarrow$ u_F only appears as Πu_F, i.e. we can replace F_h^k by F_h^{k-1}.

order $k-1$ dofs globally $\rightsquigarrow$ order $k+1$ L^2-error (HDG superconvergence)

[11] C.L. & J. Schöberl, High order exactly divergence-free HDG Methods for incompressible... flows, 2016, CMAME

The benefits of Hybrid formulations

Why to prefer HDG over DG

- element-wise assembly
- less couplings / static condensation $\rightsquigarrow$ less non-zero entries / less global dof
- postprocessing / "projected jumps" $\rightsquigarrow$ even less global dofs (diffusion dom.)
- (better suited for iterative solution)[12]

When to prefer DG over HDG

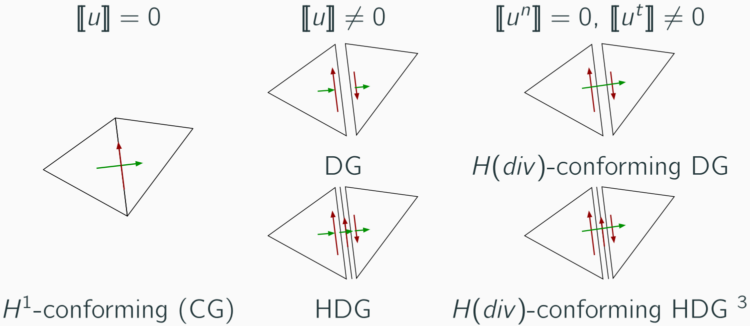
- benefits are essentially related to linear systems

The benefits of HDG vanish for explicit time discretizations

[12] C.L. & J. Schöberl, Domain Decomposition Preconditioning for High Order HDG Methods on Tetrahedral Meshes, ..., Springer, 2012

$H(div)$-conforming Finite elements with (H)DG

DG or HDG formulations are easily generalized to the vector-valued case:

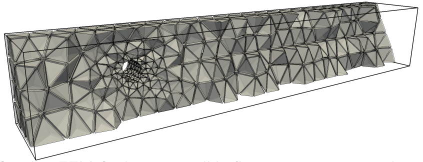


$H(div)$-FEM requires DG/HDG formulation for tangential direction!

[3] C.L., Hybrid DG methods for solving incompr. flow problems, RWTH Aachen, 2010

Comparison of sparsity ($H(\mathrm{div})$-conf. Vector Laplace)

		DG	HDG	PHDG		DG	HDG	PHDG
	#dof[K]	23	69	**38**		67	158	**112**
$k = 1$	#cdof[K]	23	69	**38**	$k = 2$	67	137	**91**
	#nzeA[K]	732	2 037	**637**		5 073	8 103	**3 628**
	#dof[K]	146	298	**237**		271	500	**423**
$k = 3$	#cdof[K]	146	229	**168**	$k = 4$	271	343	**267**
	#nzeA[K]	21 686	22 443	**12 124**		69 525	50 412	**30 588**
	#dof[K]	453	773	**681**		702	1 128	**1 022**
$k = 5$	#cdof[K]	453	480	**389**	$k = 6$	702	640	**533**
	#nzeA[K]	184 035	98 696	**64 816**		424 764	175 321	**121 944**



DG for convection

Convection problem $\operatorname{div}(wu) = f$ **(**$\operatorname{div}(w) = 0$**)**

After partial integration on each element we get

$$\int_\Omega \operatorname{div}(wu)\; v\; dx = \sum_T \left\{ -\int_T wu\; \nabla v\; dx + \int_{\partial T} w_n \hat{u} v\; ds \right\}$$

Upwind choice: $\hat{u} = u_{nb}$ if $w_n \leq 0$ (inflow), $\hat{u} = u$ if $w_n > 0$ (outflow).

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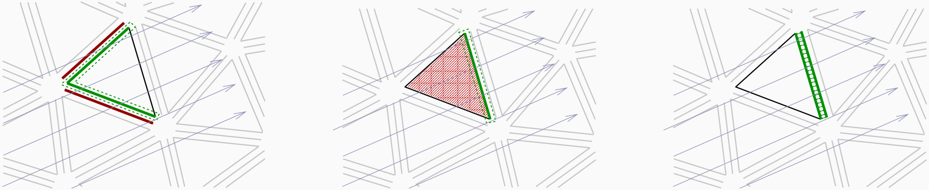

outflow stabilization: $+\sum_T \int_{\partial T_{out}} w_n (u_F - u) v_F\ ds$

[13] H. Egger & J. Schöberl, A hybrid mixed DG FEM for convection-diffusion problems , IMA Journal of Numerical Analysis, 2009

Operator-splitting time integration

Operator-splitting time integration

Semi-discrete problem

$$\begin{cases} \mathbf{M}\frac{\partial u}{\partial t} + \mathbf{A}\mathbf{u} + \mathbf{C}(\mathbf{u})\,\mathbf{u} \quad + \mathbf{B}\,\mathbf{p} = \mathbf{f} & \text{in } [0,T], \\ \mathbf{B}^T\mathbf{u} = 0 & \text{in } [0,T], \\ \mathbf{u}(t\!=\!0) = \mathbf{u}_0. \end{cases}$$

Time integration

- convection: Fully implicit schemes are very expensive due to nonlinear term $\mathbf{C}(\mathbf{u})$
- viscosity: Fully explicit schemes require small time steps (parabolic CFL)
- incompressibility: Algebraic constraint requires implicit treatment

[11] C.L. & J. Schöberl, High order exactly divergence-free HDG Methods for incompressible... flows, 2016, CMAME

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Time integration

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Operator-splitting time integration

Simplified notation

$$\mathbf{M}^* \frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}^* \mathbf{U} + \mathbf{C}^*(\mathbf{U})\ \mathbf{U} = \mathbf{F} \text{ in } \ [0, T],$$

with $\mathbf{M}^*$, $\mathbf{A}^*$, $\mathbf{C}^*$ and $\mathbf{F}$ acting on $\Sigma_h \times F_h \times Q_h$ and $\mathbf{U} \in \Sigma_h \times F_h \times Q_h$.

Building blocks for convection-diffusion type operator splitting:

Stokes-Brinkmann (unsteady Stokes):[11]

($\rightsquigarrow$ H(div)-HDG)

$$\mathbf{M}^* \frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} + \mathbf{A}^* \mathbf{U}^{n+1} = \mathbf{F}$$

$$\mathbf{U} \in \Sigma_h \times F_h \times Q_h$$

Linear hyperbolic problems:

($\rightsquigarrow$Standard DG)

$$\mathbf{M} \frac{\partial \mathbf{W}}{\partial t} = -\mathbf{C}(\bar{\mathbf{U}})\mathbf{W} \text{ or } \mathbf{C}(\bar{\mathbf{U}})\mathbf{U}$$

$$\mathbf{W} \in V_h \supset \Sigma_h \text{ (full DG)}$$

[11] C.L. & J. Schöberl, High order exactly divergence-free HDG Methods for incompressible... flows, 2016, CMAME

Classes of operator splitting (main classes)

Simplest example: Forward-Backward / Semi-Implicit Euler

$$\left(\frac{1}{\Delta t}\mathbf{M}^* + \mathbf{A}^*\right)\mathbf{U}^{n+1} = \mathbf{F}^{n+1} + \left(\frac{1}{\Delta t}\mathbf{M}^* + \mathbf{C}^*(\mathbf{U}^n)\right)\mathbf{U}^n$$

Additive splittings[14] (Combine expl./impl. schemes at compatible stages)

(+) avoids implicit solutions with $C(u)$

(+) consistent

($\sim$) same spat. treatment for expl./impl. part

(—) time steps expl./impl. are coupled.

Multiplicative splittings[15] (Sequence of expl./impl. problems)

(+) avoids implicit solutions with $C(u)$

(+) time steps expl./impl. are decoupled.

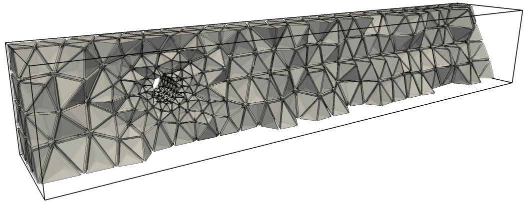
(+) diff. spat. treatments of expl./impl. part

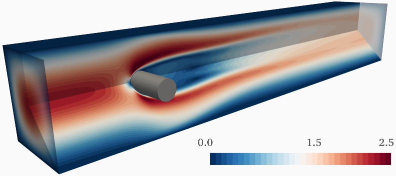
(—) add. consistency error (splitting error)

[14] U. Ascher, S. Ruuth, B. Wetton, Implicit-explicit methods for time-dependent partial differential equations, SINUM, 1995

[15] Y.Maday, A. Patera & E. Rønquist, An Operator-Integration-Factor Splitting Method for [...], J. Sci. Comp., 1990

Applications



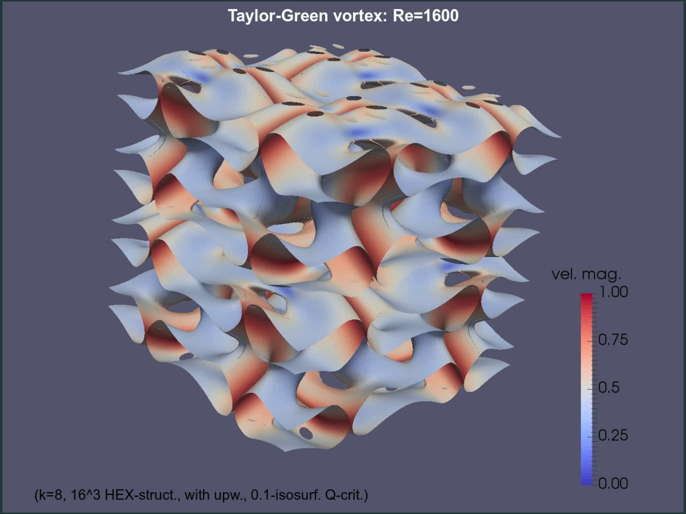


	#ndof [K]	max c_D	max c_L	min c_L	Δt [s]	comp. time [s]
$k = 3$	343	3.29331	0.00277	-0.01110	0.0080	964 × 24
$k = 4$	595	3.29853	0.00278	-0.01076	0.0040	3 087 × 24
$k = 5$	939	3.29798	0.00278	-0.01105	0.0040	6 670 × 24
ref.[16]	11 432	3.2963	0.0028	-0.01099	0.01	35 550 × 24
(Q_2/P_1^{disc})	89 760	3.2978	0.0028	-0.01100	0.005	214 473 × 48

[11] C.L. & J. Schöberl, High order exactly divergence-free HDG Methods for incompressible... flows, 2016, CMAME

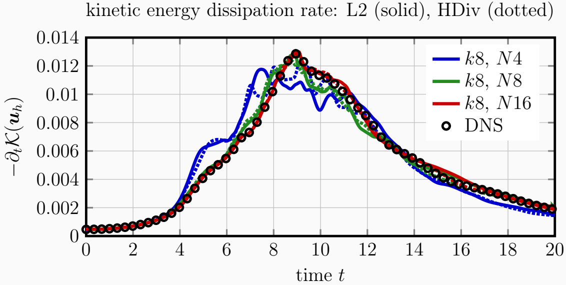
[16] E. Bayraktar & S. Turek, Benchmark Computations of 3D ... Flow ... with CFX, OpenFOAM and FeatFlow, IJCSE, 2012

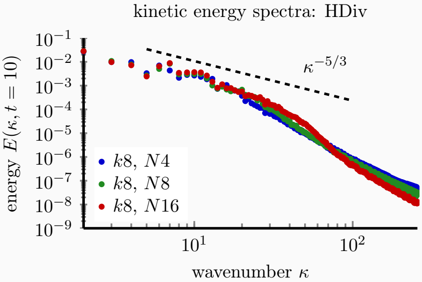
Decaying 3D turbulence ($Re = 1600$, periodic boundary conditions)



Implicit LES without overly dissipative stabilizations **(no-model LES)**

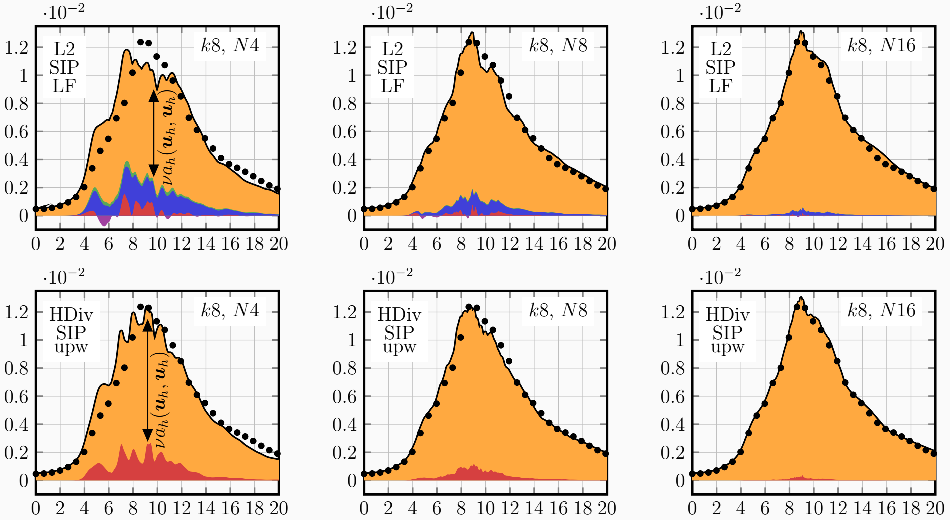
- only numerical dissipation in Upwinding and interior penalty
- no need for more for energy-stability





[17] P.W. Schroeder, N. Fehn, M. Kronbichler, C.L., G. Lube, High-order DG solvers for under-resolved turbulent incompressible flows: A comparison of L^2 and $H(\mathbf{div})$ methods, IJNMF, 2019

Taylor-Green Vortex ($Re = 1600$, periodic boundary conditions)

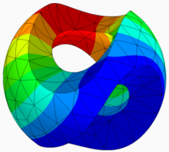


Summary, extensions & conclusion

Summary & extensions

Summary

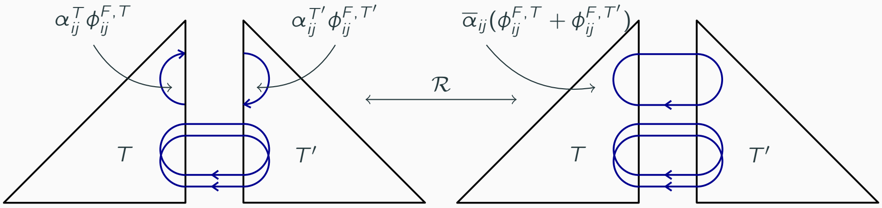
- discretely divergence-free velocities are worth the effort ($H(\mathrm{div}) \rightsquigarrow$DG)
- Weak tangential continuity: Prefer HDG over DG (linear systems)
- Tweaks: Reduced $H(\mathrm{div})$-basis, projected jumps (HDG superconvergence)
- Operator-splitting time integration (explicit schemes: DG instead of HDG)
- Implemented in `NGSolve` (`www.ngsolve.org`)



Summary & extensions

Extensions

- Further improvement: Relaxed normal-continuity while keeping benefits[18]
  (reduce facet order for normal-component to $k-1 \rightsquigarrow$ more efficient linear solvers)



[18] P.Lederer, C.L., J.Schöberl, HDG with relaxed H(div)-conformity for incompr. flows. Part I / Part II, SINUM 2018 / ESAIM:M2AN 2019

[19] G.Fu, C.L., A strongly conservative HDG[..] FEM for the coupling of Stokes and Darcy flow, Journal of Sci. Comp., 2018

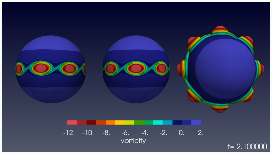
[20] P. Lederer, C.L., J. Schöberl, Divergence-free tangential FEM for incompressible flows on surfaces, (accepted in IJNME), arXiv:1909.06229

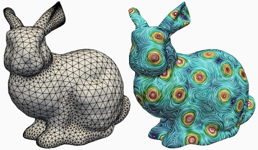
[21] G.Fu, A. Linke, C.L., T. Streckenbach, Locking free and gradient robust H(div)-conf. HDG .. linear elasticity, arXiv:2001.08610, 2020

Summary & extensions

Extensions

- Further improvement: Relaxed normal-continuity while keeping benefits[18]
- Further applications of $H(\mathrm{div})$-conforming FEM:
  - $H(div)$-HDG for Darcy-Stokes[19],
  - Surface-Navier-Stokes[20]
    (exactly tang. ($H(\mathrm{div}_\Gamma)$-conf.) FE, HDG)
  - linear elasticity[21]
    (volume-locking-free and gradient-robust)





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Outlook

Ongoing

- How far can we come without explicit turbulence modeling?
- Wall models into discretizations (wall boundary conditions/enrichment/...)
- $H(\text{div})$-conforming geometrically unfitted discretizations

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Thank you for your attention!

Back-Up Slides

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construction of $H(\mathrm{div})$-conf. FE space

hybrid mixed methods

more on operator splitting

HDG preconditioning

Realization of $H(\mathrm{div})$-conf. solutions

Re-semi-robustness

stationary Navier-Stokes

Exploiting the a priori knowledge[0] $(\mathrm{div}(u) = 0)$

$$\begin{array}{ccccccc} H^1 & \xrightarrow{\nabla} & H(\mathrm{curl}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div}) & \xrightarrow{\mathrm{div}} & L^2 \\ \cup & & \cup & & \cup & & \cup \\ V_h & \xrightarrow{\nabla} & W_h & \xrightarrow{\mathrm{curl}} & \Sigma_h & \xrightarrow{\mathrm{div}} & Q_h \end{array}$$

$1 + 3$ $\quad$ $3 + 3$ $\quad$ $3 + 1$ $\quad$ 1

Natural separation of the space (for high order)

$$\begin{aligned} V_h &= V_{\mathcal{L}_1} \\ W_h &= W_{\mathcal{N}_0} \\ \Sigma_h &= \Sigma_{\mathcal{RT}_0} \\ Q_h &= Q_{\mathcal{P}_0} \end{aligned}$$

[0] J. Schöberl, S. Zaglmayr, High order Nédélec elements with local complete sequence properties, COMPEL, 2005

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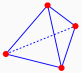
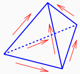
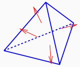
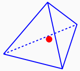

$$
1+3+a \qquad a+3+3+b \qquad b+3+1+c \qquad c+1
$$

Natural separation of the space (for high order)

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V_h &= V_{\mathcal{L}_1} + \mathrm{span}\{\varphi^V_{h.o.}\} \\
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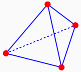 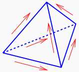 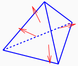 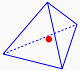

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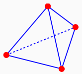
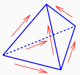
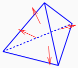
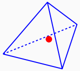

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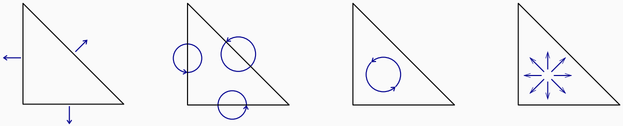
Sketch of velocity space in 2D



	RT$_0$ DOF		higher order edge DOF		higher order div.-free DOF		higher order DOF with nonz. div
	RT$_0$ shape fncs.		**curl** of H^1-edge fncs.		**curl** of H^1-el. fncs.		remainder
$\operatorname{div}(\Sigma_h^k) =$	$\bigoplus_T \mathcal{P}^0(T)$	$\oplus$	$\{0\}$	$\oplus$	$\{0\}$	$\oplus$	$\bigoplus_T [\mathcal{P}^{k-1} \cap \mathcal{P}^{0\perp}](T)$
#DOF=	3	$+$	$3k$	$+$	$\frac{1}{2}k(k-1)$	$+$	$\frac{1}{2}k(k+1)-1$

Discrete functions have only piecewise constant divergence

$\Rightarrow$ only piecewise constant pressure necessary for exact incompressibility

$$div(\Sigma_h^*) = Q_h^* = Q_{\mathcal{P}_0}$$

Mixed formulations

Mixed formulation of the Poisson problem:

Set $\sigma = -\nabla u$ to get a system of first order PDEs:

$$\left\{ \begin{array}{rcrcl} \sigma & + & \nabla u & = & 0 \\ \operatorname{div}(\sigma) & & & = & f \end{array} \right. \text{ in } \Omega + b.c.$$

with weak solutions $\sigma \in H(\operatorname{div}, \Omega)$ and $u \in L^2(\Omega)$:

$$\left\{ \begin{array}{rcrcll} (\sigma, \tau)_\Omega & - & (u, \operatorname{div}(\tau))_\Omega & = & (-u_D, \tau_n)_{\partial\Omega} & \forall\, \tau \in H(\operatorname{div}, \Omega) \\ -(\operatorname{div}(\sigma), v)_\Omega & & & = & (-f, v)_\Omega & \forall\, v \in L^2(\Omega) \end{array} \right.$$

Mixed formulations

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[0] F. Brezzi & M. Fortin, Mixed and Hybrid Finite Element Methods, 1991, Springer

(Hybrid) Mixed formulations

Mixed formulation of the Poisson problem[0]:

$$\begin{cases} (\sigma,\tau)_\Omega - (u,\operatorname{div}(\tau))_\Omega = (-u_D,\tau_n)_{\partial\Omega} & \forall\, \tau \in \Sigma_h^{k+1} \subset H(\operatorname{div},\Omega) \\ -(\operatorname{div}(\sigma),v)_\Omega = (-f,v)_\Omega & \forall\, v \in V_h^{k,\text{disc}} \subset L^2(\Omega) \end{cases}$$

Hybrid mixed discretization[0]:

Discrete spaces (discontinuous) + element-wise part. int. + constraint / lag. mult.:

$$\begin{cases} \sum_T(\sigma,\tau)_T - \sum_T(u,\operatorname{div}(\tau))_T + \underline{\sum_T(u_F,\tau_n)_{\partial T}} = 0, & \forall\, \tau \in \Sigma_h^{k+1,\underline{\text{disc}}} \\ \sum_T -(\operatorname{div}(\sigma),v)_T = (-f,v)_\Omega, & \forall\, v \in V_h^{k,\text{disc}} \\ \underline{\sum_T(\sigma_n,v_F)_{\partial T}} = 0, & \forall\, v_F \in \underline{F_h^k} \end{cases}$$

σ, u can be statically condensated: s.p.d. system for u_F.

[0] F. Brezzi & M. Fortin, Mixed and Hybrid Finite Element Methods, 1991, Springer

(Hybrid) Mixed (DG) formulations

Mixed formulation of the Poisson problem[0]:

$$\begin{cases} (\sigma, \tau)_\Omega \quad - \quad (u, \operatorname{div}(\tau))_\Omega \quad = \quad (-u_D, \tau_n)_{\partial\Omega} & \forall\, \tau \in \Sigma_h^{k+1} \subset H(\operatorname{div}, \Omega) \\ -(\operatorname{div}(\sigma), v)_\Omega \quad = \quad (-f, v)_\Omega & \forall\, v \in V_h^{k,\text{disc}} \subset L^2(\Omega) \end{cases}$$

DG (mixed):[0]

Discretization + element-wise partial integration + flux choices:

$$\begin{cases} \sum_T (\sigma, \tau)_T \quad + \sum_T (\nabla u, \tau)_T + \sum_T (\hat{u} - u, \tau_n)_{\partial T} = 0, & \forall\, \tau \in [V_h^{k-1,\text{disc}}]^d \\ \sum_T (\sigma, \nabla v)_T - \sum_T (\hat{\sigma}_n, v)_{\partial T} = (-f, v)_\Omega, & \forall\, v \in V_h^{k,\text{disc}} \end{cases}$$

Numerical trace/flux (choice): $\hat{u} = ?$, $\hat{\sigma}_n = ?$

Eliminating σ into second equation $\Longrightarrow$ primal DG

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[0] D. Arnold, F. Brezzi, B. Cockburn, LD. Marini, Unified Analysis of DG ... Elliptic Probl., SIAM J.Num. Anal., 2002

[0] F. Brezzi & M. Fortin, Mixed and Hybrid Finite Element Methods, 1991, Springer

(Hybrid) Mixed (DG) formulations

Mixed formulation of the Poisson problem[0]:

$$\left\{\begin{array}{rcll} (\sigma,\tau)_\Omega - (u,\mathrm{div}(\tau))_\Omega &=& (-u_D,\tau_n)_{\partial\Omega} & \forall\, \tau \in \Sigma_h^{k+1} \subset H(\mathrm{div},\Omega) \\ -(\mathrm{div}(\sigma),v)_\Omega &=& (-f,v)_\Omega & \forall\, v \in V_h^{k,\mathrm{disc}} \subset L^2(\Omega) \end{array}\right.$$

Hybrid DG (mixed):[0]

Discretization + element-wise partial integration + flux choices + constraint:

$$\left\{\begin{array}{ll} \sum_T(\sigma,\tau)_T + \sum_T(\nabla u,\tau)_T + \sum_T(\hat{u}-u,\tau_n)_{\partial T} = 0, & \forall\, \tau \in [V_h^{k-1,\mathrm{disc}}]^d \\ \sum_T(\sigma,\nabla v)_T - \sum_T(\hat{\sigma}_n,v)_{\partial T} = (-f,v)_\Omega, & \forall\, v \in V_h^{k,\mathrm{disc}} \\ \sum_T(\hat{\sigma}_n,v_F)_{\partial T} = 0, & \forall\, v_F \in F_h^k \end{array}\right.$$

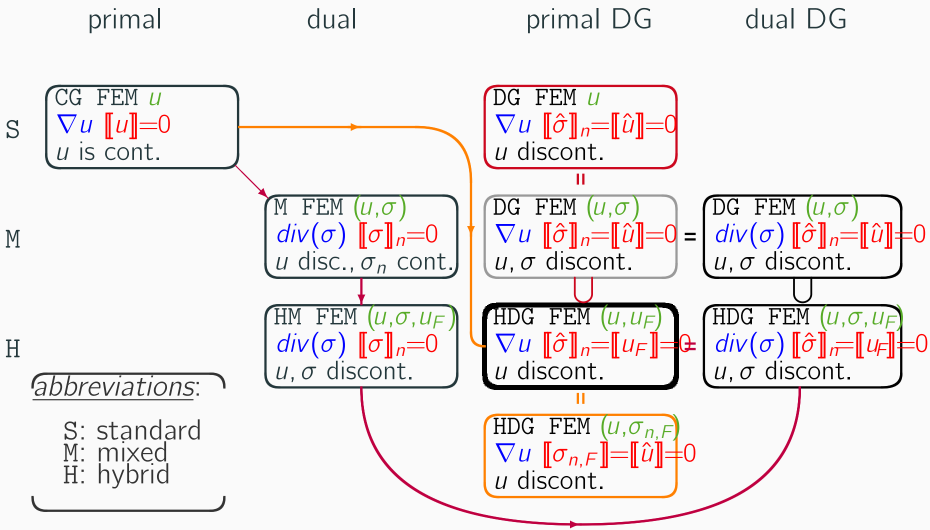
Numerical trace/flux (choice): $\hat{u} = u_F$, $\hat{\sigma}_n == -\partial_n u + \gamma_h [\![u]\!]$

Eliminating σ into second equation $\Longrightarrow$ primal Hybrid IP DG

[0] D. Arnold, F. Brezzi, B. Cockburn, LD. Marini, Unified Analysis of DG ... Elliptic Probl., SIAM J.Num. Anal., 2002

[0] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified Hybridization of DG, ... , SIAM J.Num. Anal., 2009

[0] F. Brezzi & M. Fortin, Mixed and Hybrid Finite Element Methods, 1991, Springer


primal
dual
primal DG
dual DG
S
M
H
CG FEM u
∇u $[\![u]\!]=0$
u is cont.
DG FEM u
∇u $[\![\hat{\sigma}]\!]_n=[\![\hat{u}]\!]=0$
u discont.
M FEM (u,σ)
$div(\sigma)$ $[\![\sigma]\!]_n=0$
u disc., σ_n cont.
DG FEM (u,σ)
∇u $[\![\hat{\sigma}]\!]_n=[\![\hat{u}]\!]=0$
u,σ discont.
DG FEM (u,σ)
$div(\sigma)$ $[\![\hat{\sigma}]\!]_n=[\![\hat{u}]\!]=0$
u,σ discont.
HM FEM (u,σ,u_F)
$div(\sigma)$ $[\![\sigma]\!]_n=0$
u,σ discont.
HDG FEM (u,u_F)
∇u $[\![\hat{\sigma}]\!]_n=[\![u_F]\!]=0$
u discont.
HDG FEM (u,σ,u_F)
$div(\sigma)$ $[\![\hat{\sigma}]\!]_n=[\![u_F]\!]=0$
u,σ discont.
HDG FEM $(u,\sigma_{n,F})$
∇u $[\![\sigma_{n,F}]\!]=[\![\hat{u}]\!]=0$
u discont.
abbreviations:
S: standard
M: mixed
H: hybrid

Operator splitting: Additive splitting methods

Forward-Backward / Semi-Implicit Euler:

$$\left(\frac{1}{\Delta t}\mathbf{M}^* + \mathbf{A}^*\right)\mathbf{U}^{n+1} = \mathbf{F}^{n+1} + \left(\frac{1}{\Delta t}\mathbf{M}^* + \mathbf{C}^*(\mathbf{U}^n)\right)\mathbf{U}^n$$

Structure

- Evaluate explicit and implicit parts at different time stages
- Evaluate explicit part only at old (known) time stages

[0] U. Ascher, S. Ruuth, B. Wetton, Implicit-explicit methods for time-dependent partial differential equations, SINUM, 1995

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Names/Aliases

- IMEX[0] (implicit-explicit)
- ARK (additive Runge-Kutta)
- Semi-Implicit (Euler/BDF/...)
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Dis-/Advantages of additive splitting

(+) avoids implicit solutions with $C(u)$

(+) consistent

(-) time steps for explicit and implicit are not decoupled.

[0] U. Ascher, S. Ruuth, B. Wetton, Implicit-explicit methods for time-dependent partial differential equations, SINUM, 1995

Operator splitting: Multiplicative splitting methods

Can't we decompose the problem into subproblems of the following form?

$$\left(\frac{\partial}{\partial t}\mathbf{M}^* + \mathbf{A}^*\right)\mathbf{U} = \tilde{\mathbf{F}} \quad \text{and} \quad \left(\frac{\partial}{\partial t}\mathbf{M}^* + \mathbf{C}^*(\mathbf{U})\right)\mathbf{U} = \tilde{\mathbf{F}}$$

[0] Y.Maday, A. Patera & E. Rønquist, An Operator-Integration-Factor Splitting Method for Time-Dependent Problems: ..., J. Sci. Comp., 1990

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Operator-Integration-Factor Splitting[0]

Rewrite original problem as $\frac{\partial}{\partial t}(\mathbf{Q}^{t\to t^*}\mathbf{U}) = \mathbf{Q}^{t\to t^*}\mathbf{M}^{*-1}(\mathbf{F} - \mathbf{A}^*\mathbf{U})$

with $\mathbf{Q}$ the *propagation operator*, s.t. $\mathbf{Q}^{t_1\to t_2}\mathbf{U}_1 = \mathbf{V}(t_2)$

where $\mathbf{V}$ is the solution of the explicit *propagation problem*:

$$\frac{\partial}{\partial s}\mathbf{V} = -\mathbf{M}^{*-1}\mathbf{C}^*(\mathbf{V})\,\mathbf{V}, \quad \mathbf{V}(t_1) = \mathbf{U}_1$$

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First order version

1. Propagate: $\mathbf{V}(t^n) = \mathbf{U}^n$ with $\frac{\partial}{\partial s}\mathbf{V} = -\mathbf{M}^{*-1}C(\mathbf{V}^*)\mathbf{V}, \quad \longrightarrow \bar{\mathbf{U}}^n = \mathbf{V}(t^{n+1})$.
2. Solve: $(\mathbf{M}^* + \Delta t\mathbf{A}^*)\mathbf{U}^{n+1} = \bar{\mathbf{U}}^n + \Delta t\mathbf{F}^{n+1}$

[0] Y.Maday, A. Patera & E. Rønquist, An Operator-Integration-Factor Splitting Method for Time-Dependent Problems: ..., J. Sci. Comp., 1990

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Back-Up Slides

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2. Solve: $(\mathbf{M}^* + \Delta t\mathbf{A}^*)\mathbf{U}^{n+1} = \bar{\mathbf{U}}^n + \Delta t\mathbf{F}^{n+1}$

Dis-/Advantages of additive splitting

(+) avoids implicit solutions with $C(u)$

(+) time steps for explicit and implicit are decoupled.

(—) introduces an additional consistency error (splitting error)

[0] Y.Maday, A. Patera & E. Rønquist, An Operator-Integration-Factor Splitting Method for Time-Dependent Problems: ..., J. Sci. Comp., 1990

Classes of operator splitting II

"Pseudo-implicit schemes" / fractional step ($\mathbf{F} = 0$)

1. "Stokes implicit" (saddle point, solve linear system, HDG):

$$(\mathbf{M}^* + \theta \Delta t \mathbf{A}^*)\, \mathbf{U}^1 = \left(\mathbf{M}^* - \theta \Delta t \mathbf{C}^*(\mathbf{U}^0)\right) \mathbf{U}^0$$

2. "convection implicit" (linear hyperbolic, solved by pseudo-time stepping, DG):

$$(\mathbf{M}^* + \theta^* \Delta t \mathbf{C}^*(\mathbf{U}^*))\, \mathbf{U}^2 = -\theta^* \Delta t \mathbf{A}^* \mathbf{U}^1$$

   $\mathbf{U}^*$: extrapolated (divergence-free) velocity

3. "Stokes implicit" (saddle point, solve linear system, HDG):

$$(\mathbf{M}^* + \theta \Delta t \mathbf{A}^*)\, \mathbf{U}^3 = \left(\mathbf{M}^* - \theta \Delta t \mathbf{C}^*(\mathbf{U}^2)\right) \mathbf{U}^2$$

(+) avoids implicit solutions with $\mathbf{C}^*(\cdot) + \mathbf{A}^*$

(+) 2nd order accurate, consistent to stationary solutions

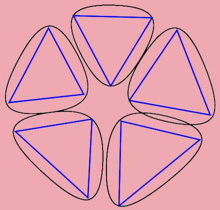
(+) time steps for explicit and implicit are decoupled.

(+) allows for different spatial treatment of implicit/explicit part

(—) stability (theory) not clear

Preconditioning (high order, domain decomposition)

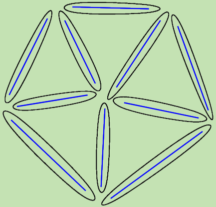
DG preconditioning:



element-by-element Schwarz preconditioner plus coarse grid[0]:

LDG/IP (proven/exp): $\kappa(C^{-1}A) \simeq k^2$

HDG preconditioning:



facet-by-facet Schwarz preconditioner plus coarse grid[0]:

H-IP (proven): $\kappa(C^{-1}A) \simeq k$

H-BR (proven): $\kappa(C^{-1}A) \simeq (\log k)^3$

H-IP/BR (exp): $\kappa(C^{-1}A) \simeq (\log k)^2$

[0] P. F. Antonietti, P. Houston, A Class of DD Preconditioners for hp-[DGFEM], J. Sci. Comput., 2011

[0] C.L. & J. Schöberl, Domain Decomposition Preconditioning for High Order HDG Methods on Tetrahedral Meshes, ..., Springer, 2012

Preconditioning (backup)

For $\lambda \in \mathcal{P}^k(F)$ define the semi-norm and norm (Schur-complement norm)

$$|\lambda|_F^2 := \inf_{u \in \mathcal{P}^k} \left\{ \|\nabla u\|_{L^2(T)}^2 + \|u - \lambda\|_{j,F}^2 \right\}$$
$$\|\lambda\|_{F,0}^2 := \inf_{u \in \mathcal{P}^k} \left\{ \|\nabla u\|_{L^2(T)}^2 + \|u - \lambda\|_{j,F}^2 + \|u - 0\|_{j,\partial T \setminus F}^2 \right\}$$

For $\lambda \in \mathcal{P}^k(F)$ with $\int_F \lambda = 0$ there holds

$$\|\lambda\|_{F,0}^2 \lesssim (\log k)^\gamma |\lambda|_F \quad \text{with } \gamma = 3$$

[0] P. F. Antonietti, P. Houston, A Class of DD Preconditioners for hp-[DGFEM], J. Sci. Comput., 2011

[0] C.L. & J. Schöberl, Domain Decomposition Preconditioning for High Order HDG Methods on Tetrahedral Meshes, ..., Springer, 2012

Ways to achieve $H(div)$-conformity and $\text{div}(\Sigma_h) \subset Q_h$

H^1 conforming FE spaces

- macro-elements: barycentric refined grids
- sufficient high order $k \geq 2d$

Scott/Vogelius 1985, Vogelius 1983, ...

$H(div)$ conforming DG spaces

- abandon H^1-conformity
- use BDM (on simplices) or RT (on quads/hexes)

Cockburn/Kanschat/Schötzau 2005, ...

DG FE space with constraints

- use standard DG spaces
- enforce normal-continuity through Lagrange mult.
- equivalent to $H(div)$ conforming FE space

Montlaur/Fernandez–Mendez/Peraire/Huerta 2009, Rhebergen/Wells 2017, Fu 2018, ...

DG FE space with penalties

- use standard DG spaces
- normal-continuity and div.-free through penalties
- converges to $H(div)$ conforming solution

Kronbichler/Wall/Fehn 2017, ...

Ways to achieve $H(div)$-conformity and $\text{div}(\Sigma_h) \subset Q_h$

Back-Up Slides

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Kronbichler/Wall/Fehn 2017, ...

Re-semi-robustness of time-dependent Nav. Stokes[0,0]

$$\frac{1}{2}\|u-u_h\|^2_{L^\infty(L^2)} + \int_0^T \nu C_\sigma |||u-u_h|||^2 + \|u-u_h\|^2_{\text{upw}} d\tau$$

$$\lesssim h^{2k} \mathbf{e}^{G_u(T)} \int_0^T S(h\|\partial_t u\|_{H^{k+1}}, \|u\|_{H^{k+1}}, \|u\|_{L^\infty}, \|\nabla u\|_{L^\infty}) d\tau$$

with $\mathbf{G}_\mathrm{u}(\mathbf{T}) = T + \|u\|_{L^1(L^\infty)} + C\|\nabla u\|_{L^1(L^\infty)}$, $S(\dots)$ independent of Re.

[0] P.Schroeder, C.L., A. Linke, G. Lube, Towards comp. flows and robust estimates [...] to time-dependent N.-S., subm. to SeMA, 2017

2D steady Nav. Stokes with $H(\text{div})$-HDG (Kovasznay)