Divergence-free tangential finite element methods for incompressible flows on surfaces

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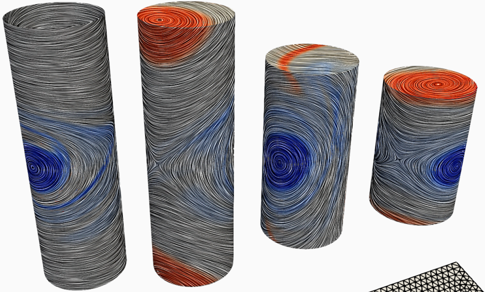
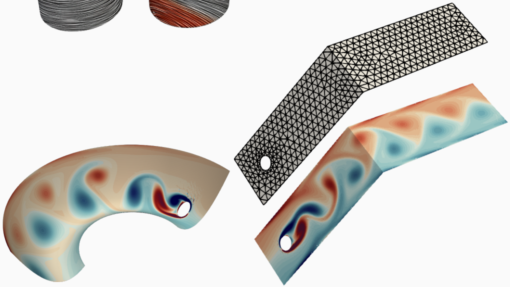
https://bit.ly/2mrrTWI

(slides)

Incompressible flows on surface



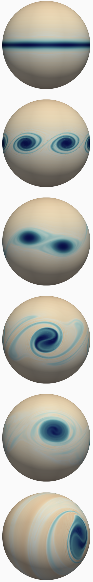


Problem classes:

- Incompressible flows on surfaces:
  - Vector Laplacian
  - Surface Stokes
  - Surface Navier–Stokes

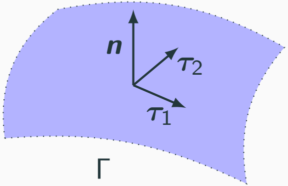
Discretization goals:

- Tangential FE spaces
- Surface divergence-conforming FEM



Some notation: Surface differential operators

Γ piecewise smooth, connected, 2D,
stationary, oriented surface embedded in $\mathbb{R}^3$.



Surface gradient (scalar ϕ):

$$\nabla_\Gamma \phi(x) = \sum_{i=1,2} \frac{\partial \phi}{\partial \boldsymbol{\tau}_i} \tau_i = P(x) \nabla \phi^e(x),$$

$P(x) = I - \boldsymbol{n} \otimes \boldsymbol{n}$, $\quad (\cdot)^e$: smooth extension

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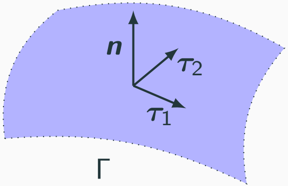

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Covariant derivative ($u = \sum_{j=1,2} u_j \boldsymbol{\tau}_j$):

$$\nabla_\Gamma u(x) = \sum_{i,j=1,2} \frac{\partial u_j}{\partial \boldsymbol{\tau}_i} \boldsymbol{\tau}_j \otimes \boldsymbol{\tau}_i$$
$$= P(x) \nabla u^e(x) P(x),$$

Surface divergence:

$$\text{div}_\Gamma = \text{tr}(\nabla_\Gamma u)$$

Surface strain tensor:

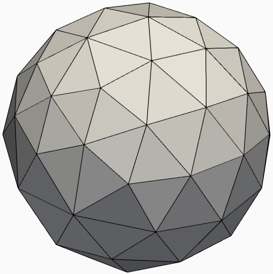
$$\varepsilon_\Gamma(u) = \tfrac{1}{2}(\nabla_\Gamma u + \nabla_\Gamma u^T)$$

First problem: Vector Laplacian

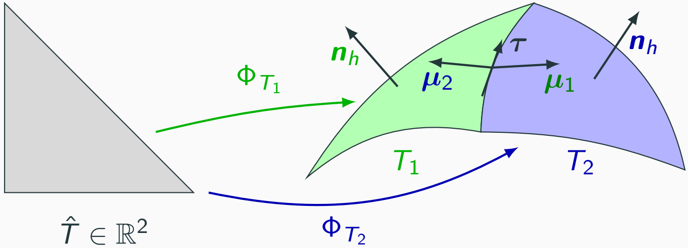
Find $u : \Gamma \to \mathbb{R}^3$ with $u \cdot \boldsymbol{n} = 0$, s.t.

$$-P \operatorname{div}_\Gamma(\varepsilon_\Gamma(u)) + u = f \quad \text{in } \Gamma,$$
$$u = 0 \quad \text{on } \partial\Gamma.$$

with $f \cdot \boldsymbol{n} = 0$.



Discrete surfaces



Discrete surface $\Gamma_h = \bigcup_{T \in \mathcal{T}_h} T$ is only C^0. $T = \Phi_T(\hat{T})$ with $\Phi_T : \mathbb{R}^2 \ni \hat{T} \to T \in \mathbb{R}^3$.

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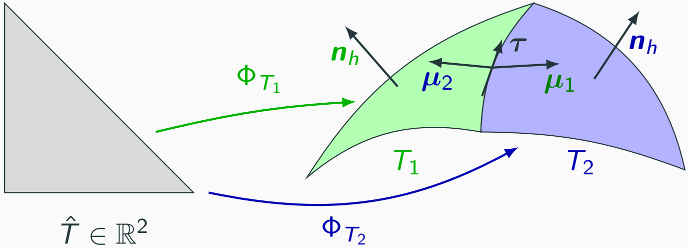


Discrete surface $\Gamma_h = \bigcup_{T \in \mathcal{T}_h} T$ is only C^0. $T = \Phi_T(\hat{T})$ with $\Phi_T : \mathbb{R}^2 \ni \hat{T} \to T \in \mathbb{R}^3$.
$\rightsquigarrow$ Discontinuous normal field, i.e. $\boldsymbol{\mu}_1 \neq -\boldsymbol{\mu}_2$ on edges $E \in \mathcal{F}_h$.

Solutions in $H^1_T(\Gamma_h)$:

- Tangential: $u \cdot \boldsymbol{n}_h = 0$
- Tangential continuity: $u_1 \cdot \boldsymbol{\tau} = u_2 \cdot \boldsymbol{\tau}$ and
- (Co-)normal continuity: $u_1 \cdot \boldsymbol{\mu}_1 + u_2 \cdot \boldsymbol{\mu}_2 = 0$.

Vectorial finite elements on surfaces

H^1-conforming approaches

Hansbo/Larson/Larsson 2016, Reuther/Voigt 2018, Fries 2018
Olshanskii/Yushutin 2018, Jankuhn/Reusken 2019

- Look for 3D solution field in surface FE space

$$u_h \in [S_h^k]^3, \;\; S_h^k := \{v_h \in H^1(\Gamma) \,|\, v_h|_T \circ \Phi_T \in \mathbb{P}^k(\hat{T}) \; \forall T \in \mathcal{T}_h\}, \;\; [S_h^k]^3 \subset [H^1(\Gamma)]^3 \supset H^1_T(\Gamma)$$

- and implement $u_h \cdot \boldsymbol{n}_h = 0$ through variational formulation:
  - Lagrange multiplier formulation
  - Penalty formulation

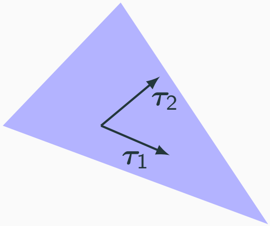
Example formulation of the Vector-Laplacian (penalty):

Find $u_h \in [S_h^k]^3$, s.t. for all $v_h \in [S_h^k]^3$ there holds

$$\int_{\Gamma_h} \varepsilon_\Gamma(u_h) : \varepsilon_\Gamma(v_h) \; dx + \int_{\Gamma_h} u_h \cdot v_h \; dx + \int_{\Gamma_h} \rho \; (u_h \cdot \boldsymbol{n}_h)(v_h \cdot \boldsymbol{n}_h) dx = \int_{\Gamma_h} f v_h \; dx$$

Our approach: abandoning H^1-conformity

Discontinuous tangential FE space



Simple tang. FE space (local coords. $\boldsymbol{\tau}_i = \partial_{\hat{x}_i}\Phi_T / \|\partial_{\hat{x}_i}\Phi_T\|$):

$$\tilde{W}_h := \{v_h \in L^2_T(\Gamma) \mid v_h|_T = v_1 \cdot \boldsymbol{\tau}_1 + v_2 \cdot \boldsymbol{\tau}_2,\ v_i \circ \Phi_T \in \mathbb{P}^k(\hat{T})\ \forall T \in \mathcal{T}_h\}$$

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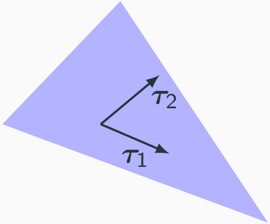

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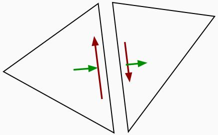
Tangential FE space by Piola transformation:

$$W_h := \{v_h \in L^2_T(\Gamma) \mid v_h|_T = \underbrace{\tfrac{1}{J_T} F_T}_{\mathcal{P}_T} \underbrace{v_h \circ \Phi_T^{-1}}_{\hat{v}_h},\ \hat{v}_h \in [\mathbb{P}^k(\hat{T})]^2\ \forall T \in \mathcal{T}_h\}$$

$$\text{with} \quad F_T = \Phi'_T(\hat{x}) = \begin{pmatrix} | & | \\ c_1\boldsymbol{\tau}_1 & c_2\boldsymbol{\tau}_2 \\ | & | \end{pmatrix} \quad \text{and} \quad J_T = \sqrt{\det(F_T^T F_T)}, \quad T \in \mathcal{T}_h.$$

$\rightsquigarrow$ tangential FE spaces are easy to construct when leaving H^1-conformity!

DG formulation for Vector Laplacian



Interior penalty formulation of second order term
Averages and jumps:

$$\{\!\{\sigma\boldsymbol{\mu}\}\!\} := \frac{\sigma|_{T_1}\boldsymbol{\mu}_1 - \sigma|_{T_2}\boldsymbol{\mu}_2}{2} \quad \text{and} \quad [\![u]\!] := [\![u]\!]_{\boldsymbol{\tau}}\boldsymbol{\tau} + [\![u]\!]_{\boldsymbol{\mu}}\overline{\boldsymbol{\mu}}$$

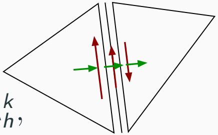
$$[\![u]\!]_{\boldsymbol{\mu}} := u|_{T_1} \cdot \boldsymbol{\mu}_1 + u|_{T_2} \cdot \boldsymbol{\mu}_2, \qquad [\![u]\!]_{\boldsymbol{\tau}} := (u|_{T_1} - u|_{T_2}) \cdot \boldsymbol{\tau}, \qquad \overline{\boldsymbol{\mu}} = \tfrac{1}{2}\boldsymbol{\mu}_1 + \tfrac{1}{2}\boldsymbol{\mu}_2.$$

DG bilinear form:

$$-\operatorname{div}(\varepsilon_\Gamma(u)) \rightsquigarrow a_h(u_h, v_h) := \sum_{T\in\mathcal{T}_h}\int_T \varepsilon_\Gamma(u_h) : \varepsilon_\Gamma(v_h)\,dx + \sum_{E\in\mathcal{F}_h}\int_E \{\!\{-\varepsilon_\Gamma(u_h)\boldsymbol{\mu}\}\!\} : [\![v_h]\!]\,ds$$
$$+ \int_E \{\!\{-\varepsilon_\Gamma(v_h)\boldsymbol{\mu}\}\!\} : [\![u_h]\!]\,ds + \frac{\alpha k^2}{h}\int_E [\![u_h]\!] : [\![v_h]\!]\,ds.$$

Hybrid DG formulation for Vector Laplacian

The facet approximation (two-sided view)



$$\boldsymbol{\lambda}_{T_1} := \lambda_{\boldsymbol{\tau}}\boldsymbol{\tau} + \lambda_{\boldsymbol{\mu}}\boldsymbol{\mu}_1 \quad \text{and} \quad \boldsymbol{\lambda}_{T_2} := \lambda_{\boldsymbol{\tau}}\boldsymbol{\tau} + \lambda_{\boldsymbol{\mu}}\boldsymbol{\mu}_2 \quad \text{for } \lambda_{\boldsymbol{\tau}}, \lambda_{\boldsymbol{\mu}} \in \Lambda_h^k,$$

with Λ_h^k the space of **scalar**, discontinuous, piecewise polynomials on the skeleton.

HDG jumps (for one element):

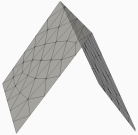
$$[\![u]\!]^H := [\![u]\!]^H_{\boldsymbol{\tau}}\boldsymbol{\tau} + [\![u]\!]^H_{\boldsymbol{\mu}}\boldsymbol{\mu}, \qquad [\![u]\!]^H_{\boldsymbol{\mu}} := u|_T \cdot \boldsymbol{\mu} - \lambda_{\boldsymbol{\mu}}, \qquad [\![u]\!]^H_{\boldsymbol{\tau}} := u|_T \cdot \boldsymbol{\tau} - \lambda_{\boldsymbol{\tau}}$$

HDG bilinear form: (allows for static condensation)

$$-\operatorname{div}(\varepsilon_\Gamma(u)) \rightsquigarrow a_h(u_h, v_h) := \sum_{T\in\mathcal{T}_h} \int_T \varepsilon_\Gamma(u_h) : \varepsilon_\Gamma(v_h)\, dx + \int_{\partial T} (-\varepsilon_\Gamma(u_h)\boldsymbol{\mu}) \cdot [\![v_h]\!]^H\, ds$$

$$+ \int_{\partial T} (-\varepsilon_\Gamma(v_h)\boldsymbol{\mu}) \cdot [\![u_h]\!]^H\, ds + \frac{\alpha k^2}{h} \int_{\partial T} [\![u_h]\!]^H \cdot [\![v_h]\!]^H\, ds.$$

A first example: house of cards (only piecewise smooth)

		
surface	Γ_1	$\Gamma_2 = \Psi(\Gamma_1),\ \Psi :$ isometry
exact solution	u^1	$u^2 = \mathcal{P}_\Psi(u^1) = \frac{1}{J}\Psi'(u^1 \circ \Psi^{-1})$

$\|\nabla_\Gamma(u^e - u_h)\|_{\Gamma_h}$
($\texttt{H1-L/P}$, $k = 3$)

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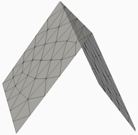
($\texttt{HDG}$, $k = 3$)



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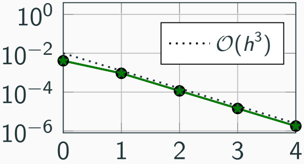

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		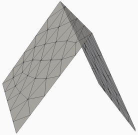
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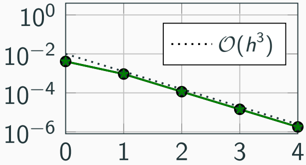
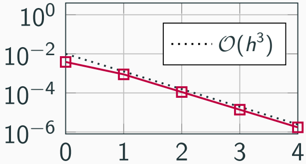
$\Vert\nabla_\Gamma(u^e - u_h)\Vert_{\Gamma_h}$ ($\texttt{H1-L/P}$, $k = 3$)		
$\Vert\nabla_\Gamma(u^e - u_h)\Vert_{\Gamma_h}$ ($\texttt{HDG}$, $k = 3$)		

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		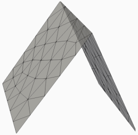
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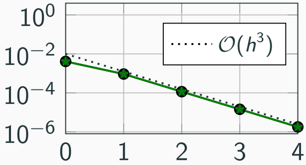
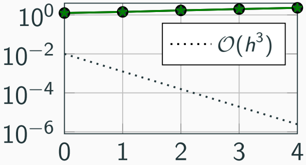
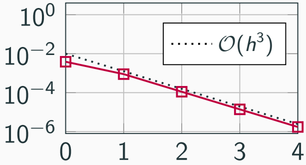
$\lVert\nabla_\Gamma(u^e - u_h)\rVert_{\Gamma_h}$ (H1-L/P, $k = 3$)		
$\lVert\nabla_\Gamma(u^e - u_h)\rVert_{\Gamma_h}$ (HDG, $k = 3$)		

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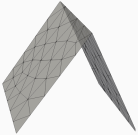

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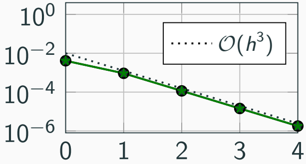

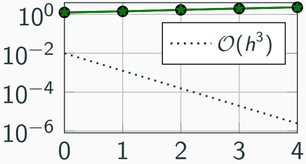

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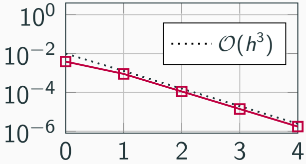

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		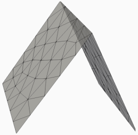	
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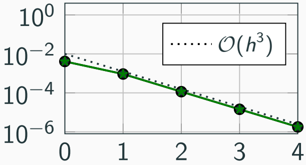
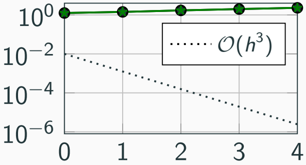
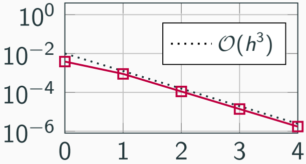
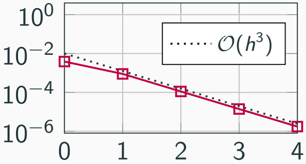
$\lVert\nabla_\Gamma(u^e - u_h)\rVert_{\Gamma_h}$ (H1-L/P, $k = 3$)			$u^2 \notin [H^1(\Gamma)]^3$
$\lVert\nabla_\Gamma(u^e - u_h)\rVert_{\Gamma_h}$ (HDG, $k = 3$)			

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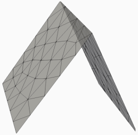

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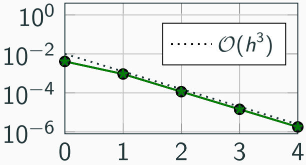

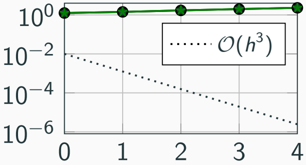

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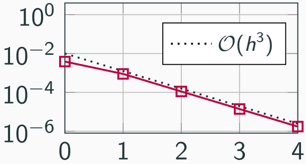
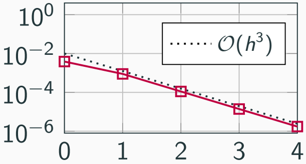




$u_h^2 = \mathcal{P}_\Psi(u_h^1)$

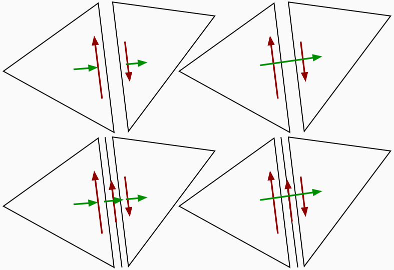
Normal-continuity

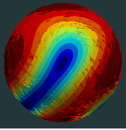
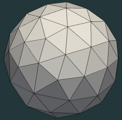
Piola transformation preserves normal moments $\rightsquigarrow$ We can easily construct $H(\mathrm{div}_\Gamma)$-conforming finite elements:

$$V_h^k := W_h^k \cap \{[\![u]\!]_{\boldsymbol{\mu}} = 0 \ \forall \ E \in \mathcal{F}_h\}$$

Consequences for (H)DG discretizations:

- For DG: $[\![u]\!] = [\![u]\!]_{\boldsymbol{\tau}} \boldsymbol{\tau}$
- For HDG: we can remove $\lambda_{\boldsymbol{\mu}}$ and $[\![u]\!]^H = [\![u]\!]_{\boldsymbol{\tau}}^H \boldsymbol{\tau}$



Vector-Laplacian on the sphere: Comparison of methods

details

H^1-conforming 3D approximation vs. HDG 2D approximation

- `dofs`:
  `Hdiv-DG` less than `H1` for $k \geq 4$
- `gdofs/nze`:
  `(Hdiv-)HDG` cheapest for higher order FEM ($k \geq 3, 4$)
- With tweaks (“projected jumps” / superconvergence):
  `(Hdiv-)HDG` beats `H1-L/P` for all orders.

⇝ Advantages in structure properties and computational overhead

⇝ Consider only `Hdiv-HDG` in the following

Unsteady surface Navier–Stokes

Find $u : \Gamma \times (0, T] \to \mathbb{R}^3$ with $u \cdot \boldsymbol{n} = 0$ on Γ and $p : \Gamma \times (0, T] \to \mathbb{R}$ s.t.

$$\begin{aligned} \partial_t u - 2\nu P \operatorname{div}_\Gamma(\varepsilon_\Gamma(u)) + (u \cdot \nabla_\Gamma) u + \nabla_\Gamma p &= f \qquad \text{on} \quad \Gamma,\ t \in (0, T], \\ \operatorname{div}_\Gamma(u) &= 0 \qquad \text{on} \quad \Gamma,\ t \in (0, T]. \end{aligned}$$

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$$\mathrm{div}_\Gamma(u) = 0 \qquad \text{on} \quad \Gamma,\ t \in (0,T].$$

Components of the discretization

- Velocity-pressure coupling: $\ldots + \nabla_\Gamma p, \qquad \mathrm{div}_\Gamma(u) = 0$
  $u_h \in V_h^k \subset H(\mathrm{div}_\Gamma),\ p_h \in Q_h = S_h^{k-1} = \mathrm{div}_\Gamma(V_h)$

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- Convection: $\ldots + (u \cdot \nabla_\Gamma) u + \ldots$

  Upwinding

Incompressible flows on surfaces

Unsteady surface Navier–Stokes

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- Convection: $\quad \dots + (u \cdot \nabla_\Gamma) u + \dots$

  Upwinding
- Time-stepping: $\quad \partial_t u + \dots$

  Operator-splitting

Benefits of $H(\mathrm{div}_\Gamma)$-conformity (inherited from flat case)

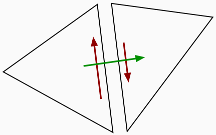
- local mass conservation (independent of geometry approx.)
  - With $u_h = \mathcal{P}_T(\hat{u}_h)$, $q_h = \hat{q}_h \circ \Phi_T^{-1}$:

$$b_h(u_h, q_h) := \sum_T \int_T \mathrm{div}_\Gamma(u_h) q_h \, dx \overset{\mathcal{P}_T}{=} \sum_T \int_{\hat{T}} |J| \frac{1}{J} \mathrm{div}_\Gamma(\hat{u}_h) \hat{q}_h \, d\hat{x} = 0 \quad \forall q_h \in Q_h,$$

$$\hat{q}_h = \mathrm{sgn}(J)\, \mathrm{div}_\Gamma(\hat{u}_h) \Longrightarrow \int_T \mathrm{div}_\Gamma(\hat{u}_h)^2 \, d\hat{x} = 0 \; \forall T \in \mathcal{T}_h,$$

$$\Longrightarrow \mathrm{div}_\Gamma(\hat{u}_h) = 0 \Longrightarrow \mathrm{div}_\Gamma(u_h) = 0 \text{ pointwise}.$$

  - normal-continuous (what leaves one element, enters another) :



Benefits of $H(\mathrm{div}_\Gamma)$-conformity (inherited from flat case)

- local mass conservation (independent of geometry approx.)
- pressure robustness: $f = g + \nabla_\Gamma \phi$ (Stokes)

$$\begin{aligned} a_h(u_h, v_h) + b_h(v_h, p_h) &= (g + \nabla_\Gamma \phi, v_h) \quad \forall v_h \in V_h, \\ b_h(u_h, q_h) &= \qquad 0 \qquad\quad \forall q_h \in Q_h. \end{aligned}$$

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$u_h \in V_h^0$, the div.-free subsp. of V_h.

$$a_h(u_h,v_h^0) \underbrace{+ b_h(v_h^0,p_h)}_{=0} = (g + \underbrace{\nabla_\Gamma \phi, v_h^0}_{=0}) \quad \forall v_h^0 \in V_h^0$$

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u __and u_h__ only depend on g, ϕ is balanced by p_h.

$$\|u - u_h\|_V = F(u)$$

Benefits of $H(\mathrm{div}_\Gamma)$-conformity (inherited from flat case)

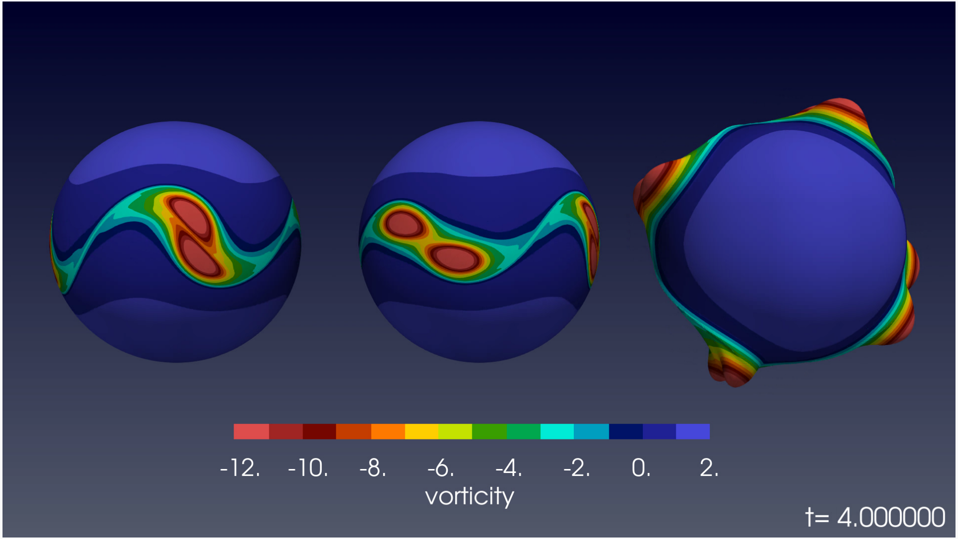
- local mass conservation (independent of geometry approx.)
- pressure robustness
- energy stability

Symmetric testing ($v_h = u_h,\ q_h = -p_h$):

$$\frac{1}{2}\partial_t(u_h, u_h) + \underbrace{a_h(u_h, u_h)}_{\geq 0} + \underbrace{c_h(u_h; u_h, u_h)}_{?} = (f, u_h)$$

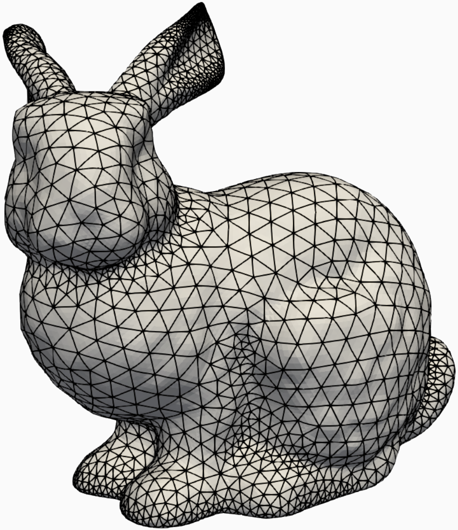
- $c_h(u_h; u_h, u_h) \geq 0$ for all established discretizations
- details depend on discretization (CG/DG/Upw./..)
- Crucial: u_h is div.free (and normal continuous)

Kelvin-Helmholtz on a sphere



⇝ https://youtu.be/pgsmzRgG2Ek

Self-organization on the Stanford bunny



⇝ https://youtu.be/gxNFa5kIdoA

Conclusion

Construction of tangential and div_Γ-conforming FEM

- H^1 -conf. $\to$ DG: tangential FEM obtained from Piola
- Piola $\rightsquigarrow$ $H(\mathrm{div}_\Gamma)$-conf.($[\![u]\!]_\mu = 0$) $\rightsquigarrow$ div-free, robust
- allows for piecewise smooth and complex surfaces
- discretization is invariant under isometries
- competitive to H^1-conf. methods (HDG) (2D $\leftrightarrow$ 3D)
- Implementation in NGSolve (`http://ngsolve.org`)
- Further properties and variants from flat case:
    - low overall dissipation (upwinding is not even necessary)
    - tweaks (red. FE spaces / "proj. jumps" / rel. $H(\mathrm{div}_\Gamma)$-conf.)

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Divergence-free tangential FEM [...] on surfaces

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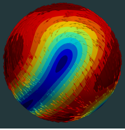
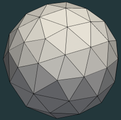
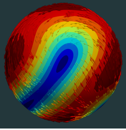
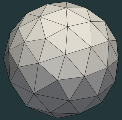
Thank you for your attention!

Backup-Slides

Vector-Laplacian on the sphere: Comparison of methods

	dof				
k	1	2	3	4	5
DG	193.5K	387.1K	645.1K	967.7K	1.4M
HDG	387.1K	677.4K	1M	1.5M	1.9M
Hdiv-DG	96.8K	241.9K	451.6K	725.8K	1.1M
Hdiv-HDG	193.5K	387.1K	645.1K	967.7K	1.4M
H1-L	64.5K	258.1K	580.6K	1M	1.6M
H1-P	48.4K	193.5K	435.5K	774.2K	1.2M

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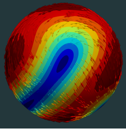
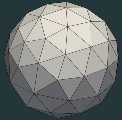
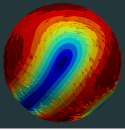
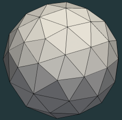
H1-L	64.5K	258.1K	451.6K	645.1K	838.7K
H1-P	48.4K	193.5K	338.7K	483.8K	629K

Vector-Laplacian on the sphere: Comparison of methods

	nze				
k	1	2	3	4	5
DG	4.6M	18.6M	51.6M	116.1M	227.6M
HDG	3.9M	8.7M	15.5M	24.2M	34.8M
Hdiv-DG	2.5M	12M	36.9M	88.3M	180.6M
Hdiv-HDG	3.9M	8.7M	15.5M	24.2M	34.8M
H1-L	1.7M	11.1M	29.7M	48.1M	88.5M
H1-P	1M	6.7M	16.7M	31.1M	49.8M

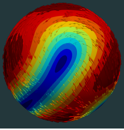
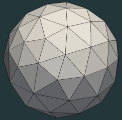
Vector-Laplacian on the sphere: Comparison of methods

Projected jumps

Reduce facet space by one order and (L^2) project jumps into $\mathbb{P}^{k-1}(E)$

$$[\![u]\!]_{\boldsymbol{\mu}}^{H} \rightsquigarrow \Pi_E^{k-1}(u|_T \cdot \boldsymbol{\mu}) - \lambda_{\boldsymbol{\mu}}, \qquad [\![u]\!]_{\boldsymbol{\tau}}^{H} \rightsquigarrow \Pi_E^{k-1} u|_T \cdot \boldsymbol{\tau} - \lambda_{\boldsymbol{\tau}}$$

Reduces the number of globally coupled `dof`.

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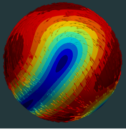 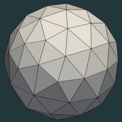

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